Free groupoids with the identity $(xy)y = yx$

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Abstract

A construction of free groupoids in the variety $\mathcal{V}$ defined by Stein identity $(xy)y = yx$ ([1]) is given. The construction is canonical i.e. the free groupoid is constructed with universe a subset of the absolutely free groupoid and suitably defined operation. In fact a countable closed set of identities that are consequences of the defining one is obtained.

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Stein identities $xy \cdot y = yx$ and $x \cdot xy = yx$, which are dual to each other, are interesting from many aspects. For each quasigroup with any of the stated identities there is an orthogonal mate which is a quasigroup as well ([1]). Stein identity together with $xy \cdot yx = x$ ($x \cdot xy = y$) define a class of cancellative groupoids with the following property: every groupoid generated by two elements has exactly four (five) elements ([5]).

In what follows $\mathcal{V}$ will denote the variety of groupoids satisfying the identity $xy \cdot y = yx$.

Let $B \neq \emptyset$ and let $\mathbf{T}_B = (T_B, \cdot)$ denote the absolutely free groupoid with a free base B. Define length $|u|$ of a term $u \in T_B$ inductively by

$$u \in B \implies |u| = 1, \qquad u = vw \implies |u| = |v| + |w|.$$

Proposition 1 *Let $(G, \cdot)$ be a $\mathcal{V}$-groupoid. Then for each $x, y, x_0, x_1, \dots \in G$ and each $n \geq 1$*

(i) $x \cdot yx = xy \cdot x$;

(ii) $x^2 \cdot x^2 = x^2$.

Proof

(i) $x \cdot yx = (yx \cdot x)x = xy \cdot x$;

(ii) $x^2 \cdot x^2 = (x^2 \cdot x) \cdot x^2 = (x \cdot x^2) \cdot x^2 = x^2 \cdot x = x^2$. $\square$

By $t(x)$ we denote a term t in which only one variable x appears. Also $t(u)$ will denote the term obtained from $t(x)$ by replacing each occurrence of x by the term u. Further on $|u|_t$ will denote the number of occurrences of u in t. The next statement is a consequence of Proposition 1 (i), (ii) and the defining identity.

Corollary 1 *If $t = t(u)$ and $t \neq u$, then $t = u^2$ is an identity in $\mathcal{V}$.* $\square$

We also need to define some notions that will abbreviate the denotation of some special forms of products. The definition is inductive.

$$x^{<0>}y = y, \; x^{<1>}y = xy, \; x^{<n+1>}y = x \cdot (x^{<n>}y)$$

Let $S = \{u \in T_B |\; u \text{ does not contain a subterm of the forms (1) - (5)}\}$.

(1) $t(u), \; u \in T_B, \; |u|_{t(u)} > 2$

(2) $((uv)^{<n>}(vu))((uv)^{<n>}u)$

(3) $((uv)^{<n>}u)((uv)^{<n>}(vu))$

(4) $((uv)^{<n+1>}u)((uv)^{<n>}(vu))$

(5) $((uv)^{<n>}(vu))((uv)^{<n+1>}u)$

where $n \geq 0, \; u, v \in T_B, \; u \neq v$. For $t, s \in S$ let

$$t * s := \begin{cases} ts & ts \in S \\ u^2 & t = t(u), \; s = s(u) \\ uv & ts = ((uv)^{<n>}(vu))((uv)^{<n>}u) \\ (uv)^{<n+1>} \cdot u & ts = ((uv)^{<n>}u)((uv)^{<n>}(vu)) \\ uv & ts = ((uv)^{<n+1>}u)((uv)^{<n>}(vu)) \\ (uv)^{<n+1>} \cdot vu & ts = ((uv)^{<n>}(vu))((uv)^{<n+1>}u) \end{cases}$$

where $n \geq 0, \; u, v \in T_B, \; u \neq v$.

We intend to prove that $(S, *)$ is a free groupoid in $\mathcal{V}$. The form (1) is a result of Corollary 1. For (2)-(5), as well as the corresponding products, one can find justification in the following property.

Proposition 2 *In any $\mathcal{V}$ - groupoid $(G, \cdot)$ the following hold:*

(a) $((uv)^{<n>}(vu))((uv)^{<n>}u) = uv, \; n \geq 0, \; u, v \in G,$

(b) $((uv)^{<n>}u)((uv)^{<n>}(vu)) = (uv)^{<n+1>} \cdot u, \; n \geq 0, \; u, v \in G,$

(c) $((uv)^{<n+1>}u)((uv)^{<n>}(vu)) = uv, \; n \geq 0, \; u, v \in G,$

(d) $((uv)^{<n>}(vu))((uv)^{<n+1>}u) = (uv)^{<n+1>} \cdot vu, \; n \geq 0, \; u, v \in G.$

Proof We prove (a), (b), (c) and (d) simultaneously by induction on n. For $n=0$, (a) is the identity $vu\cdot u = uv$; (b) is the statement (i) of Proposition 1; for (c) we use Proposition 1 (i) and the defining identity and get $(uv\cdot u)\cdot vu = (u\cdot vu)\cdot vu = (vu)u = uv$; and by similar transformations $vu\cdot(uv\cdot u) = ((uv\cdot u)\cdot vu)\cdot vu = ((u\cdot vu)\cdot vu)\cdot vu = (vu\cdot u)\cdot vu = uv\cdot vu$, i.e. (d) holds.

Assume that (a), (b), (c) and (d) hold in $(G,\cdot)$ for n. Then $((uv)^{<n+1>}(vu))((uv)^{<n+1>}u) \overset{(d),n}{=} (((uv)^{<n>}(vu))\cdot((uv)^{<n+1>}u))\cdot$ $((uv)^{<n+1>}u) = ((uv)^{<n+1>}u)\cdot((uv)^{<n>}(vu)) \overset{(c),n}{=} uv$, hence (a) holds.

Also $((uv)^{<n+1>}u)((uv)^{<n+1>}(vu)) = (((uv)^{<n+1>}(vu))\cdot((uv)^{<n+1>}u))\cdot$ $((uv)^{<n+1>}u) \overset{(a),n+1}{=} uv\cdot((uv)^{<n+1>}u) = (uv)^{<n+2>}u$, so (b) holds.

We obtain (c) in the following way $((uv)^{<n+2>}u)((uv)^{<n+1>}(vu)) \overset{(b),n+1}{=} (((uv)^{<n+1>}u)\cdot((uv)^{<n+1>}(vu)))\cdot$ $((uv)^{<n+1>}(vu)) = ((uv)^{<n+1>}(vu))\cdot((uv)^{<n+1>}u) \overset{(a),n+1}{=} uv$.

Finally, (d) holds since $((uv)^{<n+1>}(vu))((uv)^{<n+2>}u) = (((uv)^{<n+2>}u)\cdot((uv)^{<n+1>}(vu)))\cdot$ $((uv)^{<n+1>}(vu)) \overset{(c),n+1}{=} uv\cdot((uv)^{<n+1>}(vu)) = (uv)^{<n+2>}(vu)$. $\square$

Note that if uv is of the form (2) or (4) then vu is of the form (3) or (5) respectively, and vice versa. Therefore, S has the following property.

Proposition 3 *If $uv\in S$ then $vu\in S$.* $\square$

Now we can prove the main result of this note.

Theorem 1 *$(S,*)$ is a free groupoid in $\mathcal{V}$ with a free base B.*

Proof By careful examination of the forms (1) - (5) it is clear that they are mutually exclusive. Let $t,s\in S$. If ts is of form (1), then $t*s = u^2\in S$. If ts is of form (2) or (4) then uv or vu is a subterm of t hence, by Proposition 3, $uv\in S$. Finally, if ts is of form (3) or (5) then $t*s = uv\cdot t$ by definition and as before $uv, t\in S$; by checking all the forms it is clear that $uv\cdot t\in S$ where $t=(uv)^{<n>}u$ or $t=(uv)^{<n>}(vu)$ respectively. Hence, $(S,*)$ is a groupoid such that $B\subseteq S$. Also, $(S,*)$ is generated by the set B since if $uv\in S$ then $uv = u*v$ and we can use induction on the length of terms.

Next we show that $(S,*)\in\mathcal{V}$. Let $x,y\in S$. We check whether $(x*y)*y = y*x$ by distinguishing six cases for the product $x*y$.

(i) $x*y = xy$: Then, by Proposition 3, $y*x = yx$ and $xy*y = yx$ since $(xy)y = ((yx)^{<0>}(xy))\cdot((yx)^{<0>}y)$.

(ii) $x*y = u^2$, $x = x(u)$, $y = y(u)$: $(x*y)*y = u^2*y(u) = u^2 = y*x$.

(iii) $x*y = uv$, $x = (uv)^{<n>}\cdot(vu)$, $y = (uv)^{<n>}\cdot u$: $(x*y)*y = uv*y = (uv)*(uv)^{<n>}\cdot u = (uv)^{<n+1>}\cdot u = y*x$.

(iv) $x*y=(uv)^{<n+1>}\cdot u$, $x=(uv)^{<n>}\cdot u$, $y=(uv)^{<n>}\cdot(vu)$: $(x*y)*y=((uv)^{<n+1>}\cdot u)*((uv)^{<n>}\cdot(vu))=uv=y*x$.

(v) $x*y=uv$, $x=(uv)^{<n+1>}\cdot u$, $y=(uv)^{<n>}\cdot(vu)$: $(x*y)*y=uv*((uv)^{<n>}\cdot(vu))=(uv)^{<n+1>}\cdot(vu)=y*x$.

(vi) $x*y=(uv)^{<n+1>}\cdot(vu)$, $x=(uv)^{<n>}\cdot(vu)$, $y=(uv)^{<n+1>}\cdot u$: $(x*y)*y=((uv)^{<n+1>}\cdot(vu))*((uv)^{<n+1>}\cdot u)=uv=y*x$.

Finally, let $(G,\cdot)\in\mathcal{V}$ and $f:B\to G$ a mapping. Then $\hat{f}:S\to G$, defined by $\hat{f}(b)=f(b)$ for $b\in B$ and $\hat{f}(uv)=\hat{f}(u)\circ\hat{f}(v)$ for $uv\in S\setminus B$, is the desired homomorphism, by Proposition 2. $\square$

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