Weak Bisimulation for Action-Type Coalgebras [1]

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Abstract

We propose a coalgebraic definition of weak bisimulation for a class of coalgebras obtained from bifunctors in Set. Weak bisimilarity for a system is obtained as strong bisimilarity of a transformed system. The transformation consists of two steps: First, the behaviour on actions is expanded to behaviour on finite words. Second, the behaviour on finite words is taken modulo the hiding of invisible actions, yielding behaviour on equivalence classes of words closed under silent steps. The coalgebraic definition is justified by two correspondence results, one for the classical notion of weak bisimulation of Milner and another for the notion of weak bisimulation for generative probabilistic transition systems as advocated by Baier and Hermanns.

Key words: system, coalgebra, bisimulation, weak bisimulation, labelled transition system, generative probabilistic transition system

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1 Introduction

In this paper we present a definition of weak bisimulation for action type systems. A typical example of action type system is the familiar labelled transition system (LTS) (see, e.g., [Plo81,Mil90]), but also many types of probabilistic systems (see, e.g., [LS91,SL94,GSS95,BH97,Seg95]) fall into this class. In order to emphasize the role of the actions we view coalgebras as arising from bi-functors over Set.

In verification of properties of a system strong bisimilarity is often too strong an equivalence. Weak bisimilarity [Mil90] is a looser equivalence on systems that abstracts away from invisible steps. It is known that weak bisimilarity for a labelled transition system $\mathcal{S}$ amounts to strong bisimilarity on the 'double-arrowed' system $\mathcal{S}'$ induced by $\mathcal{S}$. We exploit this idea for giving a general coalgebraic definition of weak bisimilation. Our approach, given a system $\mathcal{S}$, consists of two stages.

(i) First, we define a '$*$-extension', $\mathcal{S}'$ of $\mathcal{S}$ which is a system with the same state set as $\mathcal{S}$, but with action set A^*, the set of all words over A. The system $\mathcal{S}'$ captures the behaviour of $\mathcal{S}$ on finite traces.

(ii) Next, we fix a set of invisible actions $\tau \subseteq A$ and transform $\mathcal{S}'$ into a 'weak-τ-extension' $\mathcal{S}''$ which is insensitive to τ steps. Then we define weak bisimilarity on $\mathcal{S}$ as strong bisimilarity on the weak-τ-extension $\mathcal{S}''$.

In the context of concrete probabilistic transition systems, there have been several proposals for a notion of weak bisimulation, often relying on the particular model under consideration. Segala [SL94,Seg95] proposed four notions of weak relations for his model of simple probabilistic automata. Baier and Hermanns [BH97,Bai98,BH99] have given a rather appealing definition of weak bisimulation for generative probabilistic systems. Philippou, Lee and Sokolsky [PLS00] studied weak bisimulation in the setting of the alternating model [Han91]. This work was extended to infinite systems by Desharnais, Gupta, Jagadeesan and Panangaden [DGJP02b]. The same authors also provided a metric analogue of weak bisimulation [DGJP02a].

Here, we work in a coalgebraic framework and use the general coalgebraic apparatus of bisimulation [AM89,JR96,Rut00]. For weak bisimulation in this setting, there has been early work by Rutten on weak bisimulation for while programs [Rut99] succeeded by a syntactic approach to weak bisimulation by Rothe [Rot02]. In the latter paper, weak bisimulation for a particular class of coalgebras was obtained by transforming a coalgebra into an LTS and making use of Milner's weak bisimulation there. This approach also enabled a definition of weak homomorphisms and weak simulation relations. Later, in the work of Rothe and Mašulović [RM02] a complex, but interesting coalgebraic theory was developed leading to weak bisimulation for functors that weakly preserve pullbacks. They also consider a chosen 'observer' and hidden parts of a functor. However, in the case of probabilistic and similar systems, it does not lead to intuitive results and can not be related to the concrete notions of weak bisimulation mentioned above. The so-called skip relations used in [RM02] seem to be the major obstacle as it remains unclear how quantitative information can be incorporated.

The two-phase approach of defining weak bisimilarity is, amplifying Milner's original idea, rather natural. In the category theoretical setting it has been suggested in the context of open map treatment of weak bisimulation on presheaf models [FCW99]. However, the approach taken in this paper yields a rather basic and intuitive notion of weak bisimulation. Moreover, not only for the case of labelled transition systems, but also for probabilistic systems the present coalgebraic proposal corresponds to the concrete definitions. Despite the appeal of the coalgebraic definition of weak bisimulation, proofs of correspondence result may vary from straightforward to technically involved. For example, the relevant theorem for labelled transition systems takes less than a page, whereas proving the correspondence result for generative probabilistic systems takes around 20 pages (additional machinery included).

The paper is organized as follows: In Section 2 we lay down the basic definitions and properties of the systems under consideration. Section 3 presents the definition of weak bisimulation. We show that our definition of weak bisimilarity leads to Milner's weak bisimilarity for LTSs in Section 4. Section 5 is devoted to obtaining a correspondence result for the class of generative systems of the notion of weak bisimilarity of Baier and Hermanns and our coalgebraic definition. Finally, Section 6 wraps up with some conclusions.

2 Systems and bisimilarity

We are treating systems from a coalgebraic point of view. Usually, in this context, a system is considered a coalgebra of a given Set endofunctor. For more insight in the theory of coalgebra the reader is referred to the introductory articles by Rutten, Jacobs and Gumm [Rut00,JR96,Gum99]. However, in our investigation of weak bisimilarity it is essential to explicitly specify the set of executable actions. Therefore we shall rather start from a bifunctor instead of a Set endofunctor, cf [Bor94].

A bifunctor is any functor $\mathcal{F}\colon \mathsf{Set} \times \mathsf{Set} \to \mathsf{Set}$. If $\mathcal{F}$ is a bifunctor and A is a fixed set, then a Set endofunctor $\mathcal{F}_A$ is defined by

$$\mathcal{F}_A S = \mathcal{F}(A, S), \qquad \mathcal{F}_A f = \mathcal{F}\langle id_A, f\rangle,\ f\colon S \to T. \tag{1}$$

We formulate the next proposition out of [Bor94] for further reference.

Proposition 2.1 *Let $\mathcal{F}$ be a bifunctor, and let A_1, A_2 be two fixed sets and $f : A_1 \to A_2$ a mapping. Then f induces a natural transformation η^f : $\mathcal{F}_{A_1} \Rightarrow \mathcal{F}_{A_2}$ defined by $\eta^f_S = \mathcal{F}\langle f, id_S\rangle$.* □

Definition 2.2 Let $\mathcal{F}$ be a bifunctor. If S and A are sets and α is a function, $\alpha : S \to \mathcal{F}_A(S)$, then the triple $\langle S, A, \alpha\rangle$ is called $\mathcal{F}_A$ coalgebra. A homomorphism between two $\mathcal{F}_A$-coalgebras $\langle S, A, \alpha\rangle$ and $\langle T, A, \beta\rangle$ is a function $h : S \to T$ satisfying $\mathcal{F}_A h \circ \alpha = \beta \circ h$. The $\mathcal{F}_A$-coalgebras together with their homomorphisms form a category, which we denote by $\mathsf{Coalg}^A_{\mathcal{F}}$.

An important notion in this paper is that of a bisimulation relation between two systems. We recall here the general definition of bisimulation in coalgebraic terms.

Definition 2.3 Let $\langle S, A, \alpha\rangle$ and $\langle T, A, \beta\rangle$ be two $\mathcal{F}_A$-coalgebras. A bisimulation between $\langle S, A, \alpha\rangle$ and $\langle T, A, \beta\rangle$ is a relation $R \subseteq S \times T$, such that there exists a coalgebra structure $\gamma : R \to \mathcal{F}_A R$ making the projections π_1 and π_2 coalgebra homomorphisms between the respective coalgebras, i.e. making the two squares in the following diagram commute:

$$\begin{tikzcd} S \arrow[d, "\alpha"'] & R \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] \arrow[d, dashed, "\exists\gamma"'] & T \arrow[d, "\beta"] \\ \mathcal{F}_A S & \mathcal{F}_A R \arrow[l, "\mathcal{F}_A\pi_1"] \arrow[r, "\mathcal{F}_A\pi_2"'] & \mathcal{F}_A T \end{tikzcd}$$

Two states $s \in S$ and $t \in T$ are bisimilar, notation $s \sim t$ if they are related by some bisimulation between $\langle S, A, \alpha\rangle$ and $\langle T, A, \beta\rangle$.

Let $\mathcal{F}_A$ and $\mathcal{G}_A$ be Set functors, and let $\eta : \mathcal{F}_A \Rightarrow \mathcal{G}_A$ be a natural transformation. The natural transformation η determines a functor $\mathcal{T} : \mathsf{Coalg}^A_{\mathcal{F}} \to \mathsf{Coalg}^A_{\mathcal{G}}$ defined by

$$\mathcal{T}(\langle S, A, \alpha\rangle) = \langle S, A, \eta_S \circ \alpha\rangle, \quad \mathcal{T}(f) = f. \tag{2}$$

We will refer to the functor $\mathcal{T}$ as the functor *induced* by the natural transformation η. It is known (cf. [Rut00]) that functors induced by natural transformations preserve homomorphisms and thus preserve bisimulation relations, in particular bisimilarity.

Next we present two basic types of systems, labelled transition systems and generative systems, which will be treated in more detail in Section 4 and Section 5. We give their concrete definitions first, as well as their corresponding concrete definitions of bisimulation relations, cf. [Mil89,LS91,GSS95].

Definition 2.4 A labelled transition system, or LTS for short, is a triple $\langle S, A, \to\rangle$ where S and A are sets and $\to\, \subseteq S \times A \times S$. We speak of S as the set of states, of A as the set of labels or actions the system can perform and of $\to$ as the transition relation. As usual we denote $s \xrightarrow{a} s'$ whenever $\langle s, a, s'\rangle \in\, \to$.

Definition 2.5 Let $\langle S, A, \to\rangle$ be an LTS. An equivalence relation $R \subseteq S \times S$ is a (strong) bisimulation on $\langle S, A, \to\rangle$ if and only if whenever $\langle s, t\rangle \in R$ then for all $a \in A$ the following holds:

$s \xrightarrow{a} s'$ implies that there exists $t' \in S$ with $t \xrightarrow{a} t'$ and $\langle s', t'\rangle \in R$.

Two states s and t are called bisimilar if and only if they are related by some bisimulation relation. Notation $s \sim_l t$.

When replacing the transition relation of an LTS by a "probabilistic transition relation", the so-called generative probabilistic systems are obtained.

Definition 2.6 A generative probabilistic system is a triple $\langle S, A, \mathrm{P}\rangle$ where S and A are sets and $\mathrm{P} : S \times A \times S \to [0, 1]$ with the property that for $s \in S$,

$$\sum_{a \in A,\ s' \in S} \mathrm{P}(s, a, s') \in \{0, 1\}. \tag{3}$$

We speak of S as the set of states, of A as the set of labels or actions the system can perform and of P as the probabilistic transition relation. Condition (3) states that for all $s \in S$, $\mathrm{P}(s, \_, \_)$ is either a distribution over $A \times S$ or $\mathrm{P}(s, \_, \_) \equiv 0$, i.e s is a terminating state. As usual we denote $s \overset{a[p]}{\to} s'$ whenever $\mathrm{P}(s, a, s') = p$, and $s \overset{a}{\to} s'$ for $\mathrm{P}(s, a, s') > 0$.

Remark 2.7 In order to clarify the condition (3) let us recall that the sum of an arbitrary family $\{x_i \mid i \in I\}$ of non-negative real numbers is defined as

$$\sum_{i \in I} x_i = \sup\{\sum_{i \in J} x_i \mid J \subseteq I, J \text{ finite}\}.$$

Note that, if $\sum_{i \in I} x_i < \infty$, then the set $\{x_i \mid i \in I, x_i \neq 0\}$ is at most countable.

Definition 2.8 Let $\langle S, A, \mathrm{P}\rangle$ be a generative system. An equivalence relation $R \subseteq S \times S$ is a (strong) bisimulation on $\langle S, A, \mathrm{P}\rangle$ if and only if whenever $\langle s, t\rangle \in R$ then for all $a \in A$ and for all equivalence classes $C \in S/R$

$$\mathrm{P}(s, a, C) = \mathrm{P}(t, a, C).$$

Here we have put $\mathrm{P}(s, a, C) = \sum_{s' \in C} \mathrm{P}(s, a, s')$. Two states s and t are bisimilar if and only if they are related by some bisimulation relation. Notation $s \sim_g t$.

Let us turn to the coalgebraic side. It is known that the LTSs can be viewed as coalgebras corresponding to the bifunctor

$$\mathcal{L} = \mathcal{P}(\mathcal{I}d \times \mathcal{I}d).$$

Namely, if $\langle S, A, \to\rangle$ is an LTS, then $\langle S, A, \alpha\rangle$, where $\alpha : S \to \mathcal{L}_A(S)$ is defined by

$$\langle a, s'\rangle \in \alpha(s) \iff s \overset{a}{\to} s'$$

is an $\mathcal{L}_A$ coalgebra, and vice-versa. Also, the generative systems can be considered as coalgebras corresponding to the bifunctor

$$\mathcal{G} = \mathcal{D}(\mathcal{I}d \times \mathcal{I}d) + 1.$$

Here $\mathcal{D}$ denotes the distribution functor, that is, $\mathcal{D} : \mathsf{Set} \to \mathsf{Set}$

$$\mathcal{D}X = \{\mu : X \to [0,1] \mid \textstyle\sum_{x \in X} \mu(x) = 1\}$$

$$(\mathcal{D}f)(\mu)(y) = \textstyle\sum_{f(x)=y} \mu(x), \quad f : X \to Y, \mu \in \mathcal{D}X, y \in Y.$$

If $\langle S, A, \mathrm{P} \rangle$ is a generative system, then $\langle S, A, \alpha \rangle$ is a $\mathcal{G}_A$ coalgebra where $\alpha : S \to \mathcal{G}_A(S)$ is given by

$$\alpha(s)(a, s') = \mathrm{P}(s, a, s'),$$

and vice-versa. Thereby we interpret the singleton set 1 as the set containing the zero-function on $A \times S$. Note that $\alpha(s)$ is the zero-function if and only if s is a terminating state.

In the literature it is common to restrict to generative systems $\langle S, A, \alpha \rangle$ where for any state s the function $\alpha(s)$ has finite support. However, in many respects, this restriction to generative systems with finite support is not necessary.

The concrete notion of bisimilarity for LTSs and generative systems and the respective notions of bisimilarity obtained from Definition 2.3 coincide. For the case of LTSs a direct proof was given, for example, by Rutten [Rut00]. For generative systems this fact goes back to the work of de Vink and Rutten [VR99] where Markov systems were considered, and was treated in [BSV03] for generative systems with finite support.

We describe a general procedure to obtain coincidence-results of this kind. This method already appeared implicitely in [BSV]. It applies to LTSs as well as to generative systems in their full generality. We will also use it to obtain a concrete characterization of bisimilarity for another, more complex, functor, cf. Section 5.

Definition 2.9 Let $R \subseteq S \times T$ be a relation, and $\mathcal{F}$ a Set functor. The relation R can be lifted to a relation $\equiv_{\mathcal{F},R} \subseteq \mathcal{F}S \times \mathcal{F}T$ defined by

$$x \equiv_{\mathcal{F},R} y \iff \exists z \in \mathcal{F}R\colon \mathcal{F}\pi_1(z) = x,\ \mathcal{F}\pi_2(z) = y.$$

The following lemma is obvious from Definition 2.3.

Lemma 2.10 *A relation $R \subseteq S \times T$ is a bisimulation between the $\mathcal{F}_A$ systems $\langle S, A, \alpha \rangle$ and $\langle T, A, \beta \rangle$ if and only if*

$$\langle s, t \rangle \in R \implies \alpha(s) \equiv_{\mathcal{F}_A,R} \beta(t). \tag{4}$$

□

Note that the condition (4) is commonly referred to as a transfer condition.

A functor is said to weakly preserve total pullbacks if it transforms any pullback diagram with epi legs into a weak pullback diagram. The characterization of bisimilarity will follow from the next lemma.

Lemma 2.11 *If the functor $\mathcal{F}$ weakly preserves total pullbacks and R is an equivalence on S, then $\equiv_{\mathcal{F},R}$ is the pullback in* Set *of the cospan*

$$\mathcal{F}S \xrightarrow{\mathcal{F}c} \mathcal{F}(S/R) \xleftarrow{\mathcal{F}c} \mathcal{F}S \tag{5}$$

where $c\colon S \to S/R$ is the canonical morphism mapping each element to its equivalence class.

Proof. Since R is an equivalence relation and therefore reflexive, the left diagram below is a pullback diagram with epi legs.

$$\begin{tikzcd}
 & R \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\
S \arrow[dr, "c"'] & & S \arrow[dl, "c"] \\
 & S/R &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \mathcal{F}R \arrow[dl, "\mathcal{F}\pi_1"'] \arrow[dr, "\mathcal{F}\pi_2"] & \\
\mathcal{F}S \arrow[dr, "\mathcal{F}c"'] & & \mathcal{F}S \arrow[dl, "\mathcal{F}c"] \\
 & \mathcal{F}S/R &
\end{tikzcd}$$

By the assumption, the right diagram is a weak pullback diagram. By Definition 2.9 the map $\omega\colon \mathcal{F}R \to \equiv_{\mathcal{F},R}$, $\omega(z) = \langle \mathcal{F}\pi_1(z), \mathcal{F}\pi_2(z)\rangle$, is surjective and it makes the two upper triangles of the next diagram commutative:

$$\begin{tikzcd}
 & \equiv_{\mathcal{F},R} \arrow[ddl, "\pi_1"'] \arrow[ddr, "\pi_2"] & \\
 & \mathcal{F}R \arrow[u, "\omega"'] \arrow[dl, "\mathcal{F}\pi_1"] \arrow[dr, "\mathcal{F}\pi_2"'] & \\
\mathcal{F}S \arrow[dr, "\mathcal{F}c"'] & & \mathcal{F}S \arrow[dl, "\mathcal{F}c"] \\
 & \mathcal{F}S/R &
\end{tikzcd}$$

Since ω is surjective the outer square of the above diagram also commutes, and by the existence of ω from the weak pullback $\mathcal{F}R$ to $\equiv_{\mathcal{F},R}$, $\equiv_{\mathcal{F},R}$ is a weak pullback as well. However, since it has projections as legs it is a pullback. $\Box$

Suppose that a functor $\mathcal{F}$ weakly preserves total pullbacks and assume that R is an equivalence bisimulation on S, i.e., R is both an equivalence relation and a bisimulation on S, such that $\langle s,t\rangle \in R$. The pullback in Set of the cospan (5) is the set $\{\, \langle x,y\rangle \mid \mathcal{F}c(x) = \mathcal{F}c(y)\,\}$. By Lemma 2.11 this set coincides with the lifted relation $\equiv_{\mathcal{F},R}$. Thus $x \equiv_{\mathcal{F},R} y \iff \mathcal{F}c(x) = \mathcal{F}c(y)$. Therefore, we obtain the transfer condition for the particular notion of bisimulation if we succeed in expressing concretely $(\mathcal{F}c \circ \alpha)(s) = (\mathcal{F}c \circ \alpha)(t)$ in terms of the representation of $\alpha(s)$ and $\alpha(t)$.

For example, consider the LTS functor $\mathcal{L}_A$, which preserves weak pullbacks. For $X \in \mathcal{L}_A(S)$, i.e. $X \subseteq A \times S$, we have $\mathcal{L}_A(c)(X) = \mathcal{P}\langle id_A, c\rangle(X) =$

$\langle id_A, c\rangle(X) = \{\langle a, c(s)\rangle \mid \langle a, s\rangle \in X\}$. Using Lemma 2.10 we get that an equivalence $R \subseteq S \times S$ is a coalgebraic bisimulation for an LTS $\langle S, A, \alpha\rangle$ if and only if

$$\langle s, t\rangle \in R \implies \{\, \langle a, c(s')\rangle \mid \langle a, s'\rangle \in \alpha(s) \,\} = \{\, \langle a, c(t')\rangle \mid \langle a, t'\rangle \in \alpha(t) \,\}$$

or, equivalently

$$\langle s, t\rangle \in R \implies (\ s \xrightarrow{a} s' \implies \exists t' \in S \colon t \xrightarrow{a} t' \wedge \langle s', t'\rangle \in R\ ).$$

Hence we have obtained the following property.

Lemma 2.12 *An equivalence relation R on a set S is a bisimulation on the LTS $\langle S, A, \alpha\rangle$ according to Definition 2.3 for the functor $\mathcal{L}_A$ if and only if it is a bisimulation according to Definition 2.5.* □

Often weak pullback preservation is required for the functors to be "well-behaved", for example in order that bisimilarity is an equivalence. It can easily be seen that already the weaker condition of weakly preserving total pullbacks suffices for bisimilarity to be an equivalence. We have relaxed the weak pullback preservation condition since in Section 5 we will need a bisimilarity characterization of a functor that transforms total pullbacks to weak pullbacks, but does not preserve weak pullbacks.

Next we establish the weak pullback preservation of $\mathcal{G}_A$. For the functor defining generative systems with finite support weak pullback preservation was proven by de Vink and Rutten [VR99], using the graph theoretic min cut - max flow theorem, and by Moss [Mos99], using an elementary matrix fill-in property. Following Moss [Mos99] we show that the needed matrix fill-in property can be used and holds for arbitrary, infinite, matrices as well. For the sake of completeness we give the proofs in full detail.

Lemma 2.13 *The functor $\mathcal{D}$ preserves weak pullbacks.*

Proof. It suffices to show that a pullback diagram

$$\begin{tikzcd} & P \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ X \arrow[dr, "f"'] & & Y \arrow[dl, "g"] \\ & Z & \end{tikzcd}$$

will be transformed to a weak pullback diagram (cf. [Gum99]). Let P' be the pullback of the cospan $\mathcal{D}X \xrightarrow{\mathcal{D}f} \mathcal{D}Z \xleftarrow{\mathcal{D}g} \mathcal{D}Y$. Then there exists $\gamma : \mathcal{D}P \to P'$

such that the next diagram commutes

$$\begin{tikzcd}
 & \mathcal{D}P \arrow[d, "\gamma"] \arrow[ddl, "\mathcal{D}\pi_1"'] \arrow[ddr, "\mathcal{D}\pi_2"] & \\
 & P' \arrow[dl, "\pi_1"] \arrow[dr, "\pi_2"'] & \\
\mathcal{D}X \arrow[dr, "\mathcal{D}f"'] & & \mathcal{D}Y \arrow[dl, "\mathcal{D}g"] \\
 & \mathcal{D}Z &
\end{tikzcd}$$

and it is enough to show that γ is surjective. Let $\langle u, v\rangle \in P'$ be given. if $\mu \in \mathcal{D}P$ is such that

$$(\mathcal{D}\pi_1)(\mu) = u, \qquad (\mathcal{D}\pi_2)(\mu) = v \tag{6}$$

then $\gamma(\mu) = \langle u, v\rangle$ since π_1 and π_2 are jointly injective. Hence the task is to find a function $\mu \in \mathcal{D}P$ which satisfies (6). More explicitely we have to find $\mu : P \to [0,1]$ such that for all $x_0 \in X, y_0 \in Y$

$$\sum_{y\in Y:\langle x_0,y\rangle\in P} \mu(x_0, y) = u(x_0), \qquad \sum_{x\in X:\langle x,y_0\rangle\in P} \mu(x, y_0) = v(y_0) \tag{7}$$

For if $\mu : P \to [0,1]$ satisfies (7), then $\mu \in \mathcal{D}P$ and (6) holds.

The set P can be written as the union

$$P = \bigcup_{z\in Z} f^{-1}(\{z\}) \times g^{-1}(\{z\})$$

of disjoint rectangles, in fact rectangles with non-overlapping edges. Therefore, the existence of a map μ which satisfies condition (7) is equivalent to the condition that for all $z \in Z$ there exists a function $\mu_z : f^{-1}(\{z\}) \times g^{-1}(\{z\}) \to [0,1]$ such that for all $x_0 \in f^{-1}(\{z\})$, and all $y_0 \in g^{-1}(\{z\})$,

$$\sum_{y\in g^{-1}(\{z\})} \mu_z(x_0, y) = u(x_0), \qquad \sum_{x\in f^{-1}(\{z\})} \mu_z(x, y_0) = v(y_0). \tag{8}$$

Since $\langle u, v\rangle \in P$, we have

$$\sum_{x\in f^{-1}(\{z\})} u(x) = (\mathcal{D}f)(u)(z) = (\mathcal{D}g)(v)(z) = \sum_{y\in g^{-1}(\{z\})} v(y). \tag{9}$$

Thus we may apply the following matrix-fill-in property, Lemma 2.14. $\square$

Lemma 2.14 *Let C and D be sets and let $\phi : C \to [0,1]$ and $\psi : D \to [0,1]$ be such that*

$$\sum_{x\in C} \phi(x) = \sum_{y\in D} \psi(y) < \infty \tag{10}$$

Then there exists a function $\mu : C \times D \to [0,1]$ such that for any $x_0 \in C$ and any $y_0 \in D$

$$\sum_{y \in D} \mu(x_0, y) = \phi(x_0), \qquad \sum_{x \in C} \mu(x, y_0) = \psi(y_0). \tag{11}$$

Proof. We first consider the case when both C and D are countably infinite, i.e. we take $C = D = \mathbb{N}_0$. We recursively define a function

$$F : \mathbb{N} \to \mathbb{N}_0 \times \mathbb{N}_0 \times (\mathbb{N}_0 \times \mathbb{N}_0 \to \mathbb{R})$$

where $F(n) = (k(n), l(n), \mu_n)$ as follows. Put $F(1) = \langle k(1), l(1), \mu_1 \rangle$ for $k(1) = l(1) = 0$ and $\mu_1(k,l) = 0$ for all $k, l \in \mathbb{N}_0$. Assume $F(n)$ has already been defined. Put

$$\mu_{n+1}(k,l) = \begin{cases} \mu_n(k,l) & \langle k,l \rangle \neq \langle k(n), l(n) \rangle \\ \min\{\phi(k(n)) - \sum_{l<l(n)} \mu_n(k(n),l), & \\ \qquad \psi(l(n)) - \sum_{k<k(n)} \mu_n(k,l(n))\} & \langle k,l \rangle = \langle k(n), l(n) \rangle \end{cases}$$

and

$$k(n+1) = \begin{cases} k(n)+1 & \mu_{n+1}(k(n),l(n)) = \phi(k(n)) - \sum_{l<l(n)} \mu_n(k(n),l) \\ k(n) & \text{otherwise} \end{cases}$$

$$l(n+1) = \begin{cases} l(n)+1 & \mu_{n+1}(k(n),l(n)) = \psi(l(n)) - \sum_{k<k(n)} \mu_n(k,l(n)) \\ l(n) & \text{otherwise} \end{cases}$$

It is obvious that F satisfies the following properties (12).

$$\begin{aligned} & k(n+1) + l(n+1) > k(n) + l(n),\ k(n+1) \geq k(n),\ l(n+1) \geq l(n) \\ & \mu_{n+1}(k,l) = \mu_n(k,l),\ \langle k,l \rangle \neq \langle k(n), l(n) \rangle \\ & \mu_{n+1}(k,l) = 0,\ k > k(n) \text{ or } l > l(n) \end{aligned} \tag{12}$$

We next show that F also satisfies the following properties (13).

$$\begin{aligned} & \textstyle\sum_{k \in \mathbb{N}_0} \mu_n(k, l_0) \begin{cases} = \psi(l_0) & l_0 < l(n) \\ \leq \psi(l_0) & l_0 \geq l(n) \end{cases} \\ & \textstyle\sum_{l \in \mathbb{N}_0} \mu_n(k_0, l) \begin{cases} = \phi(k_0) & k_0 < k(n) \\ \leq \phi(k_0) & k_0 \geq k(n) \end{cases} \end{aligned} \tag{13}$$

For $n = 1$ surely $l_0 \geq l(n)$ and $\sum_{k \in \mathbb{N}_0} \mu_1(k, l_0) = 0 \leq \psi(l_0)$. Assume that the conditions hold for n. If $l_0 < l(n)$ then

$$\sum_{k\in\mathbb{N}_0} \mu_{n+1}(k,l_0) = \sum_{k\in\mathbb{N}_0} \mu_n(k,l_0) = \psi(l_0).$$

If $l_0 > l(n)$, then $l_0 \geq l(n+1)$ and

$$\sum_{k\in\mathbb{N}_0} \mu_{n+1}(k,l_0) = \sum_{k\in\mathbb{N}_0} \mu_n(k,l_0) \leq \psi(l_0).$$

Finally, if $l_0 = l(n)$, then

$$\sum_{k\in\mathbb{N}_0} \mu_{n+1}(k,l_0) = \sum_{k<k(n)} \mu_n(k,l_0) + \mu_{n+1}(k(n),l(n)) + \sum_{k>k(n)} \mu_n(k,l_0).$$

By (12) the last summand vanishes and by the definition of $\mu_{n+1}(k(n),l(n))$ we have

$$\mu_{n+1}(k(n),l(n)) \leq \psi(l(n)) - \sum_{k<k(n)} \mu_n(k,l(n)) \tag{14}$$

Hence $\sum_{k\in\mathbb{N}_0} \mu_{n+1}(k,l_0) \leq \psi(l_0)$. Moreover, if $l_0 < l(n+1)$ in (14) equality holds and thus also $\sum_{k\in\mathbb{N}_0} \mu_{n+1}(k,l_0) = \psi(l_0)$. The second property of (13) follows the same way.

We next show that

$$\mu_n(k,l) \in [0,1]$$

for all n,k,l, inductively on n. For $n=1$ it is trivial. Assume that $\mu_m(k,l) \in [0,1]$ for all $m \leq n$ and $k,l \in \mathbb{N}_0$. Then also $\mu_{n+1}(k,l) \in [0,1]$ for $\langle k,l\rangle \neq \langle k(n),l(n)\rangle$. Since all $\mu_n(k,l)$ are non-negative we have

$$\mu_{n+1}(k(n),l(n)) \leq \min\{\phi(k(n)),\psi(l(n))\} \leq 1.$$

Moreover, by (13) we obtain

$$\phi(k(n)) \geq \sum_{l\in\mathbb{N}_0} \mu_n(k(n),l) \geq \sum_{l<l(n)} \mu_n(k(n),l),$$

$$\psi(l(n)) \geq \sum_{k\in\mathbb{N}_0} \mu_n(k,l(n)) \geq \sum_{k<k(n)} \mu_n(k,l(n))$$

and hence

$$0 \leq \mu_{n+1}(k(n),l(n)).$$

Since $n \mapsto \langle k(n),l(n)\rangle$ is injective, for every fixed pair $\langle k,l\rangle$, the sequence $(\mu_n(k,l))_{n\in\mathbb{N}}$ is either constantly 0, which happens if $\langle k,l\rangle \notin \{\langle k(n),l(n)\rangle \mid n\in\mathbb{N}\}$ or

$$\mu_n(k,l) = \begin{cases} 0 & n \leq n_0 \\ \mu_{n_0+1}(k,l) & n > n_0 \end{cases}$$

in case $\langle k, l\rangle = \langle k(n_0), l(n_0)\rangle$. In particular, we have established

$$\mu_n(k,l) \le \mu_{n+1}(k,l),\ n \in \mathbb{N}. \tag{15}$$

Now, we define $\mu : \mathbb{N}_0 \times \mathbb{N}_0 \to [0,1]$ by

$$\mu(k,l) = \lim_{n\to\infty} \mu_n(k,l).$$

We show that μ satisfies the properties required in the assertion of the lemma. By (12) at least one of the sequences $(k(n))_{n\in\mathbb{N}}$, $(l(n))_{n\in\mathbb{N}}$ must tend to infinity, say $k(n)$ does. Let $k_0 \in \mathbb{N}_0$ be given and let $n \in \mathbb{N}$ be such that $k_0 < k(n)$. Then for all $m \ge n$

$$\mu_n(k_0,l) = \mu_m(k_0,l) = \mu(k_0,l)$$

and thus

$$\sum_{l\in\mathbb{N}_0} \mu(k_0,l) = \sum_{l\in\mathbb{N}_0} \mu_n(k_0,l) = \phi(k_0),$$

i.e. the first part of (11) holds true. It follows that

$$\sum_{l\in\mathbb{N}_0} \psi(l) = \sum_{k\in\mathbb{N}_0} \phi(k) = \sum_{k\in\mathbb{N}_0}\sum_{l\in\mathbb{N}_0} \mu(k,l) = \sum_{l\in\mathbb{N}_0}\sum_{k\in\mathbb{N}_0} \mu(k,l), \tag{16}$$

where the change in the order of summation is justified by the fact that $\mu_n(k,l) \ge 0$. Since $\psi(l) \ge \sum_{k\in\mathbb{N}_0} \mu_n(k,l)$ for all n we obtain that

$$\sum_{k\in\mathbb{N}_0} \mu(k,l) = \lim_{n\to\infty} \sum_{k\in\mathbb{N}_0} \mu_n(k,l) \le \psi(l).$$

Hereby, the change of the limit and the sum is allowed since $\mu_n(k,l)$ is a non-negative, monotone sequence. Now (16) implies that

$$\sum_{k\in\mathbb{N}_0} \mu(k,l) = \psi(l),\ l \in \mathbb{N}_0.$$

Similarly one obtains $\sum_{l\in\mathbb{N}_0} \mu(k,l) = \phi(k),\ k \in \mathbb{N}_0$, and completes the proof in the case $C = D = \mathbb{N}_0$.

Assume now that C, D, ϕ, ψ are as in the formulation of the lemma. Consider $C' = \{x \in C \mid \phi(x) \ne 0\}$, $D' = \{x \in D \mid \psi(x) \ne 0\}$, $\phi' = \phi|_{C'}$, $\psi' = \psi|_{D'}$. Then C' and D' are at most countable. If $\mu' : C' \times D' \to [0,1]$ is such that for any $x_0 \in C', y_0 \in D'$

$$\sum_{y\in D'} \mu'(x_0,y) = \phi(x_0), \quad \sum_{x\in C'} \mu'(x,y_0) = \psi(y_0)$$

then the function $\mu : C \times D \to [0,1]$ defined by

$$\mu(x,y) = \begin{cases} \mu'(x,y) & \langle x,y \rangle \in C' \times D' \\ 0 & \text{otherwise} \end{cases}$$

fulfills the requirements of the lemma. Hence it is enough to consider the case when C and D are at most countable. Write $C = \{c_k \mid k \in \mathbb{N}_0, k < |C|\}$ and $D = \{d_l \mid l \in \mathbb{N}_0, l < |D|\}$ and define $\phi', \psi' : \mathbb{N}_0 \to [0,1]$ by

$$\phi'(k) = \begin{cases} \phi(c_k) & k < |C| \\ 0 & \text{otherwise} \end{cases} \qquad \psi'(l) = \begin{cases} \psi(d_l) & l < |D| \\ 0 & \text{otherwise} \end{cases}$$

We obtain $\mu' : \mathbb{N}_0 \times \mathbb{N}_0 \to [0,1]$ with $\sum_{l \in \mathbb{N}_0} \mu'(k_0, l) = \phi'(k_0)$ and $\sum_{k \in \mathbb{N}_0} \mu'(k, l_0) = \psi'(l_0)$ for all $k_0, l_0 \in \mathbb{N}_0$. If $k_0 \geq |C|$ then $\phi'(k_0) = 0$ and hence $\mu'(k_0, l) = 0$ for $l \in \mathbb{N}_0$. Similarly, for $l_0 \geq |D|$, $\mu'(k, l_0) = 0$ for $k \in \mathbb{N}_0$. Thus

$$\mu(c_k, d_l) = \mu'(k,l), \;\; k < |C|, \;\; l < |D|$$

satisfies the requirements of the lemma. □

Some simple derivations now suffice to show the next characterization result.

Lemma 2.15 *An equivalence relation R on a set S is a bisimulation on the generative system $\langle S, A, \alpha \rangle$ according to Definition 2.3 for the functor $\mathcal{G}_A$ if and only if it is a bisimulation according to Definition 2.8.* □

3 Weak bisimulation for action-type coalgebras

In this section we present a general definition of weak bisimulation for action-type systems. Our idea arises as a generalization of what is known from the literature for concrete types of systems. In our opinion, a weak bisimulation on a given system must be a strong bisimulation on a suitably transformed system obtained from the original one.

The given definition of weak bisimulation consists of two phases. First we define a $*$-*extended system*, that captures the behaviour of the original system when extending from the given set of actions A to A^*, the set of words over A. The $*$-extension should emerge from the original system in a faithful way. The second phase considers invisibility. Given a subset $\tau \subseteq A$ of invisible actions, we restrict the $*$-extension to visible behaviour only, by defining a so-called, *weak-τ-extended system*. Then a weak bisimulation relation on the original system is any bisimulation relation on the weak-τ-extension.

Definition 3.1 Let $\mathcal{F}$ and $\mathcal{G}$ be two bifunctors. Let Φ be a map assigning to every $\mathcal{F}_A$ coalgebra $\langle S, A, \alpha \rangle$, a $\mathcal{G}_{A^*}$ system $\langle S, A^*, \alpha' \rangle$, on the same state set, such that the following conditions are met

(1) Φ is injective, i.e. $\Phi(\langle S, A, \alpha\rangle) = \Phi(\langle S, A, \beta\rangle) \Rightarrow \alpha = \beta$;

(2) Φ preserves and reflects bisimilarity, i.e. $s \sim t$ in the system $\langle S, A, \alpha\rangle$ if and only if $s \sim t$ in the transformed system $\Phi(\langle S, A, \alpha\rangle)$.

Then Φ is called a $*$-translation and we say that $\Phi(\langle S, A, \alpha\rangle)$ is a $*$-extension of $\langle S, A, \alpha\rangle$.

The conditions (1) and (2) in Definition 3.1 make sure that the original system is "embedded" in its $*$-extension, cf. [BSV03,BSV,SV03]. The fact that a $*$-translation may lead to systems of a new type (bifunctor $\mathcal{G}$) might seem counterintuitive at first sight. However, this extra freedom is necessary since in some cases (cf. Section 5, generative systems) the starting functor is not expressive enough to allow for a $*$-extension.

A way to obtain $*$-translations follows from a previous result. Namely, if $\lambda\colon \mathcal{F}_A \Rightarrow \mathcal{G}_{A^*}$ is a natural transformation with injective components and the functor $\mathcal{F}_A$ preserves weak pullbacks, then the induced functor (see equation (2)) is a $*$-translation, cf. [BSV03, Theorem 3.9]. However, we shall see later that considering $*$-translations emerging from natural transformations is not enough, actually it does not cover known concrete cases.

Having extended an $\mathcal{F}_A$ system to its $*$-extension it is time to hide invisible actions. Let $\tau \subseteq A$. Consider the function $h_\tau\colon A^* \to (A \setminus \tau)^*$ defined inductively via specifying the function on the generators of A^* by: $h_\tau(a) = a$ if $a \notin \tau$ and $h_\tau(a) = \epsilon$ for $a \in \tau$ where ϵ denotes the empty word. The function h_τ is deleting all the occurrences of elements of τ in a word of A^*. Consider the set $A_\tau = (A \setminus \tau)^*$. By Proposition 2.1, we get the following.

Corollary 3.2 *$\eta^\tau\colon \mathcal{G}_{A^*} \Rightarrow \mathcal{G}_{A_\tau}$ given by $\eta^\tau_S = \mathcal{G}\langle h_\tau, id_S\rangle$ is a natural transformation.* □

Let Ψ_τ be the functor from $\mathsf{Coalg}_{\mathcal{G}}^{A^*}$ to $\mathsf{Coalg}_{\mathcal{G}}^{A_\tau}$ induced by the natural transformation η^τ, i.e. $\Psi_\tau(\langle S, A^*, \alpha'\rangle) = \langle S, A_\tau, \alpha''\rangle$ for $\alpha'' = \eta^\tau_S \circ \alpha'$ and $\Psi_\tau f = f$ for any morphism $f : S \to T$ (see (2)). As mentioned before, the induced functor preserves bisimilarity. The composition of a $*$-translation Φ and the hiding functor Ψ_τ we denote by $W_\tau = \Psi_\tau \circ \Phi$ and call it a weak-τ-translation. A weak-τ-translation, or equivalently, the pair $\langle \Phi, \tau\rangle$, yields a notion of weak bisimulation with respect to Φ and τ.

Definition 3.3 Let $\mathcal{F}$, $\mathcal{G}$ be two bifunctors, Φ a $*$-translation from $\mathcal{F}$ to $\mathcal{G}$ and $\tau \subseteq A$. Let $\langle S, A, \alpha\rangle$ and $\langle T, A, \beta\rangle$ be two $\mathcal{F}_A$ systems. A relation $R \subseteq S \times T$ is a weak bisimulation w.r.t $\langle \Phi, \tau\rangle$ if and only if it is a bisimulation between $W_\tau(\langle S, A, \alpha\rangle)$ and $W_\tau(\langle T, A, \beta\rangle)$. Two states $s \in S$ and $t \in T$ are weakly bisimilar w.r.t $\langle \Phi, \tau\rangle$, notation $s \approx_\tau t$, if they are related by some weak bisimulation w.r.t. $\langle \Phi, \tau\rangle$.

Next we prove that any relation $\approx_\tau$ obtained in this way, satisfies the properties that are intuitively expected from a weak bisimilarity relation.

Proposition 3.4 *Let $\mathcal{F}$, $\mathcal{G}$ be two bifunctors, Φ a $*$-translation from $\mathcal{F}$ to $\mathcal{G}$, $\langle S, A, \alpha\rangle$ an $\mathcal{F}_A$-coalgebra, $\tau \subseteq A$ and let $\approx_\tau$ denote the weak bisimilarity on $\langle S, A, \alpha\rangle$ w.r.t. $\langle \Phi, \tau\rangle$. Then the following hold:*

(i) $\sim\ \subseteq\ \approx_\tau$ *for any* $\tau \subseteq A$
i.e. strong bisimilarity implies weak.

(ii) $\sim\ =\ \approx_\emptyset$
i.e. strong bisimilarity is weak bisimilarity in absence of invisible actions.

(iii) $\tau_1 \subseteq \tau_2 \Rightarrow\ \approx_{\tau_1}\ \subseteq\ \approx_{\tau_2}$ *for any* $\tau_1, \tau_2 \subseteq A$.
i.e. when more actions are invisible the weak bisimilarity relation gets coarser.

Proof.

(i) Assume $s \sim t$ in $\langle S, A, \alpha\rangle$. Since Φ preserves bisimilarity (Definition 3.1) we have that $s \sim t$ in $\Phi(\langle S, A, \alpha\rangle)$. Next, since Ψ_τ preserves bisimilarity we get $s \sim t$ in $\Psi_\tau \circ \Phi(\langle S, A, \alpha\rangle)$, which by Definition 3.3 means $s \approx_\tau t$ in $\langle S, A, \alpha\rangle$.

(ii) From (i) we get $\sim\ \subseteq\ \approx_\emptyset$. For the opposite inclusion, note that the natural transformation $\eta^\emptyset$ from Corollary 3.2 is just the identity natural transformation. Therefore the induced functor $\Psi_\emptyset$ is just the identity functor on $\mathsf{Coalg}_{\mathcal{G}}^{A^*}$. Now assume $s \approx_\emptyset t$ in $\langle S, A, \alpha\rangle$. This means $s \sim t$ in $W_\emptyset(\langle S, A, \alpha\rangle)$, i.e. $s \sim t$ in $\Psi_\emptyset \circ \Phi(\langle S, A, \alpha\rangle)$, i.e. $s \sim t$ in $\Phi(\langle S, A, \alpha\rangle)$. Since, by Definition 3.1, every $*$-translation reflects bisimilarity we get $s \sim t$ in $\langle S, A, \alpha\rangle$.

(iii) Let $\tau_1 \subseteq \tau_2$. Consider the diagram

$$\begin{tikzcd} A^* \arrow[r, "h_{\tau_2}"] \arrow[d, "h_{\tau_1}"'] & (A \setminus \tau_2)^* \\ (A \setminus \tau_1)^* \arrow[ur, "h_{\tau_1,\tau_2}"'] & \end{tikzcd}$$

where h_{τ_1,τ_2} is the map deleting all occurrences of elements of τ_2 in a word of $(A \setminus \tau_1)^*$. The diagram commutes since first deleting all occurrences of elements of τ_1 followed by deleting all occurrences of elements of τ_2, in a word of A^* is the same as just deleting all occurrences of elements of τ_2.

Denote by η^{τ_1}, η^{τ_2}, η^{τ_1,τ_2} the natural transformations from Corollary 3.2, Proposition 2.1, corresponding to h_{τ_1}, h_{τ_2}, h_{τ_1,τ_2} respectively. They make the following diagram commute.

$$\begin{tikzcd} \mathcal{G}_{A^*} \arrow[r, Rightarrow, "\eta^{\tau_2}"] \arrow[d, Rightarrow, "\eta^{\tau_1}"'] & \mathcal{G}_{A_{\tau_2}} \\ \mathcal{G}_{A_{\tau_1}} \arrow[ur, Rightarrow, "\eta^{\tau_1,\tau_2}"'] & \end{tikzcd}$$

Since the functors Ψ_{τ_1}, Ψ_{τ_2}, Ψ_{τ_1,τ_2} are induced by the natural transformations η^{τ_1}, η^{τ_2}, η^{τ_1,τ_2}, respectively, by (2) it holds that

$$\Psi_{\tau_2} = \Psi_{\tau_1,\tau_2} \circ \Psi_{\tau_1} \tag{17}$$

and they all preserve bisimilarity. Now assume $s \approx_{\tau_1} t$ in $\langle S, A, \alpha\rangle$. This means that $s \sim t$ in the system $\Psi_{\tau_1} \circ \Phi(\langle S, A, \alpha\rangle)$. Then, since Ψ_{τ_1,τ_2} preserves bisimilarity we have $s \sim t$ in the system $\Psi_{\tau_1,\tau_2} \circ \Psi_{\tau_1} \circ \Phi(\langle S, A, \alpha\rangle)$ which by equation (17) is the system $\Psi_{\tau_2} \circ \Phi(\langle S, A, \alpha\rangle)$ and we find $s \approx_{\tau_2} t$ in $\langle S, A, \alpha\rangle$. $\square$

For further reference, we introduce some more notation. For any $w \in A_\tau$, we denote $B_w = h_\tau^{-1}(\{w\}) \subseteq A^*$. We refer to the sets B_w as blocks. Note that $B_w = \tau^* a_1 \tau^* \cdots \tau^* a_k \tau^*$ for $w = a_1 \ldots a_k \in A_\tau = (A \setminus \tau)^*$.

4 Weak bisimulation for labelled transition systems

In this section we show that in the case of LTS there exists a $*$-translation according to the general definition, such that weak bisimulation in the concrete case [Mil89] coincides with weak bisimulation induced by this $*$-translation. First we recall the definition of concrete weak bisimulation for LTSs.

Definition 4.1 Let $\langle S, A, \rightarrow\rangle$ be an LTS. Assume $\tau \in A$ is an invisible action. An equivalence relation $R \subseteq S \times S$ is a weak bisimulation on $\langle S, A, \rightarrow\rangle$ if and only if whenever $\langle s, t\rangle \in R$ then

$$s \xrightarrow{a} s' \text{ implies that there exists } t' \in S \text{ with } t \xrightarrow{\tau}{}^* \circ \xrightarrow{a} \circ \xrightarrow{\tau}{}^* t' \text{ and } \langle s', t'\rangle \in R.$$

for all $a \in A \setminus \{\tau\}$, and

$$s \xrightarrow{\tau} s' \text{ implies that there exists } t' \in S \text{ with } t \xrightarrow{\tau}{}^* t' \text{ and } \langle s', t'\rangle \in R.$$

Two states s and t are called weakly bisimilar if and only if they are related by some weak bisimulation relation. Notation $s \approx_l t$.

We now present a definition of a $*$-translation that will give us the same weak bisimilarity relation. Let $\mathcal{L}$, $\mathcal{L}_A$ be the functors for LTSs, as introduced in Section 2.

Definition 4.2 Let Φ assign to every LTS, i.e. any $\mathcal{L}_A$ coalgebra $\langle S, A, \alpha\rangle$ the $\mathcal{L}_{A^*}$ coalgebra $\langle S, A^*, \alpha'\rangle$ where for $w = a_1 \ldots a_k \in A^*$, $\langle w, s'\rangle \in \alpha'(s)$ if and only if there exist states $s_1, \ldots, s_{k-1} \in S$ such that $s \xrightarrow{a_1} s_1 \xrightarrow{a_2} s_2 \cdots s_{k-1} \xrightarrow{a_k} s'$. We use the convenient notation $s \stackrel{w}{\Rightarrow} s'$ for $\langle w, s'\rangle \in \alpha'(s)$.

Theorem 4.3 *The assignment Φ from Definition 4.2 is a $*$-translation.*

Proof. We need to prove that Φ is injective and reflects and preserves bisimilarity. Let $\Phi(\langle S, A, \alpha\rangle) = \Phi(\langle S, A, \beta\rangle) = \langle S, A^*, \alpha'\rangle$. Then $\langle a, s'\rangle \in \alpha(s) \iff \langle a, s'\rangle \in \alpha'(s) \iff \langle a, s'\rangle \in \beta(s)$. Hence for any state s, $\alpha(s) = \beta(s)$.

Let $s \sim t$ in $\Phi(\langle S, A, \alpha\rangle) = \langle S, A^*, \alpha'\rangle$. Hence there exists an equivalence bisimulation relation R such that $\langle s, t\rangle \in R$ and (according to Lemma 2.12) for all $w \in A^*$,

if $s \overset{w}{\Rightarrow} s'$ then there exists $t' \in S$ such that $t \overset{w}{\Rightarrow} t'$ and $\langle s', t'\rangle \in R$.

Assume $s \xrightarrow{a} s'$ in $\langle S, A, \alpha\rangle$, i.e., $\langle a, s'\rangle \in \alpha(s)$. Then $s \overset{a}{\Rightarrow} s'$ and therefore there exists $t' \in S$ with $\langle s', t'\rangle \in R$ and $t \overset{a}{\Rightarrow} t'$, i.e., $t \xrightarrow{a} t'$. Hence, R is a bisimulation on $\langle S, A, \alpha\rangle$ i.e. $s \sim t$ in the original system. Conversely, for the preservation, let $s \sim t$ in $\langle S, A, \alpha\rangle$ and let R be an equivalence bisimulation relation such that $\langle s, t\rangle \in R$. Assume $s \overset{w}{\Rightarrow} s'$, for some word $w = a_1 \dots a_k \in A^*$. Then there exist states $s_1, \dots, s_{k-1} \in S$ such that $s \xrightarrow{a_1} s_1 \xrightarrow{a_2} s_2 \cdots s_{k-1} \xrightarrow{a_k} s'$. By a simple inductive argument one gets that there exist states $t_1, \dots, t_{k-1}, t' \in S$ such that $t \xrightarrow{a_1} t_1 \xrightarrow{a_2} t_2 \cdots t_{k-1} \xrightarrow{a_k} t'$ where $\langle s_i, t_i\rangle \in R$ and $\langle s', t'\rangle \in R$. Hence $t \overset{w}{\Rightarrow} t'$ for $\langle s', t'\rangle \in R$, i.e, R is a bisimulation on $\langle S, A^*, \alpha'\rangle$ and $s \sim t$ holds in the $*$-extension. $\square$

Note that if Φ is a functor induced by a natural transformation η and if $\langle S, A, \alpha\rangle, \langle S, A, \beta\rangle$ are two systems such that, for some $s \in S$, $\alpha(s) = \beta(s)$, then

$$\alpha'(s) = \eta_S(\alpha(s)) = \eta_S(\beta(s)) = \beta'(s) \tag{18}$$

for $\langle S, A, \alpha'\rangle = \Phi(\langle S, A, \alpha\rangle$, $\langle S, A, \beta'\rangle = \Phi(\langle S, A, \beta\rangle$. However, the following simple example shows that the $*$-translation Φ from Definition 4.2 violates (18).

Example 4.4 Let $S = \{s_1, s_2, s_3\}$ and $A = \{a, b, c\}$. Consider the LTSs:

$$\langle S, A, \alpha\rangle : s_1 \xrightarrow{a} s_2 \xrightarrow{b} s_3 \quad \text{and} \quad \langle S, A, \beta\rangle : s_1 \xrightarrow{a} s_2 \xrightarrow{c} s_3.$$

Obviously $\alpha(s_1) = \beta(s_1)$. However, $\alpha'(s_1) = \{\langle a, s_2\rangle, \langle ab, s_3\rangle\}$ while $\beta'(s_1) = \{\langle a, s_2\rangle, \langle ac, s_3\rangle\}$.

Theorem 4.5 *Let $\langle S, A, \alpha\rangle$ be an LTS. Let $\tau \in A$ be an invisible action and $s, t \in S$ any two states. Then $s \approx_{\{\tau\}} t$ according to Definition 3.3 w.r.t the pair $\langle \Phi, \{\tau\}\rangle$ if and only if $s \approx_l t$ according to Definition 4.1.*

Proof. Assume $s \approx_{\{\tau\}} t$ for $s, t \in S$ of an LTS $\langle S, A, \alpha\rangle$. This means that $s \sim t$ in the LTS $\langle S, A_{\{\tau\}}, \eta_S^{\{\tau\}} \circ \alpha'\rangle$, i.e., there exists an equivalence bisimulation R on this system with $\langle s, t\rangle \in R$. Here, as usual, $\langle S, A^*, \alpha'\rangle = \Phi(\langle S, A, \alpha\rangle)$. Note that

$$\begin{aligned}
(\eta_S^{\{\tau\}} \circ \alpha')(s) &= \eta_S^{\{\tau\}}(\alpha'(s)) \\
&= \mathcal{P}(\langle h_{\{\tau\}}, id_S\rangle)(\alpha'(s)) \\
&= \{\langle h_{\{\tau\}}(w), s'\rangle \mid \langle w, s'\rangle \in \alpha'(s)\} \\
&= \{\langle a_1 \dots a_k, s'\rangle \mid \exists w \in \tau^* a_1 \tau^* \dots \tau^* a_k \tau^* \colon s \overset{w}{\Rightarrow} s'\}
\end{aligned}$$

We denote the transition relation of the weak-τ-system $\langle S, A_{\{\tau\}}, \eta_S^{\{\tau\}} \circ \alpha'\rangle$ by

$\Rightarrow_\tau$. The above shows that for any word $w = a_1 \dots a_k \in A_\tau$

$$s \overset{w}{\Rightarrow}_\tau s' \iff \langle w, s' \rangle \in (\eta_S^{\{\tau\}} \circ \alpha')(s) \iff \exists v \in B_w = \tau^* a_1 \tau^* \dots \tau^* a_k \tau^* \colon s \overset{v}{\Rightarrow} s'.$$

We will show that the relation R is a weak bisimulation on $\langle S, A, \alpha \rangle$ according to Definition 4.1. Let $s \xrightarrow{a} s'$ $(a \neq \tau)$. Then $s \overset{a}{\Rightarrow} s'$, implying $s \overset{a}{\Rightarrow}_\tau s'$. Since R is a bisimulation on the weak-τ-system, there exists t' such that $t \overset{a}{\Rightarrow}_\tau t'$ and $\langle s', t' \rangle \in R$. We only need to note here that $\overset{a}{\Rightarrow}_\tau = \xrightarrow{\tau}{}^* \circ \xrightarrow{a} \circ \xrightarrow{\tau}{}^*$. The case $s \xrightarrow{\tau} s'$ is analogous.

For the opposite, let R be a weak bisimulation on $\langle S, A, \alpha \rangle$ according to Definition 4.1 such that $\langle s, t \rangle \in R$. It is easy to show by induction that for all $\langle s, t \rangle \in R$ and for any $a \in A$, if $s \xrightarrow{\tau}{}^* \circ \xrightarrow{a} \circ \xrightarrow{\tau}{}^* s'$ then there exists t' such that $t \xrightarrow{\tau}{}^* \circ \xrightarrow{a} \circ \xrightarrow{\tau}{}^* t'$ and $\langle s', t' \rangle \in R$. Hence, if $s \overset{a}{\Rightarrow}_\tau s'$ then there exists t' with $t \overset{a}{\Rightarrow}_\tau t'$ and $\langle s', t' \rangle \in R$. Another simple inductive argument on k leads to the conclusion that for any word $w = a_1 \dots a_k \in A_\tau$, if $s \overset{w}{\Rightarrow}_\tau s'$ then there exists a t' such that $t \overset{w}{\Rightarrow}_\tau t'$ and $\langle s', t' \rangle \in R$, i.e. R is a bisimulation on the weak-τ-system and hence $s \approx_{\{\tau\}} t$. $\square$

5 Weak bisimulation for generative systems

In this section we deal with generative systems and their weak bisimilarity. Inspired by the existing work by Baier and Hermanns [BH97,Bai98,BH99], we provide a functor that suits for a definition of a $*$-translation for generative systems. That way we obtain a coalgebraic definition of weak bisimulation for this type of systems and at the end we show that our definition, although on first sight much stronger, coincides with the definition of Baier and Hermanns. Unlike in the case of LTSs, here the $*$-translation really leaves the class of generative systems.

This section is divided into three parts: First we introduce and establish some needed notions and properties of paths in a generative system and define a measure on the set of paths, where we basically follow the lines of Baier and Hermanns [BH99,Bai98]. In the second part we define a translation and prove that it is a $*$-translation which therefore provides us with a notion of weak-τ-bisimulation. The final part is devoted to the proof of correspondence of the notion of weak-τ-bisimulation defined by means of the given $*$-translation and the concrete notion by Baier and Hermanns.

5.1 Construction and properties of Prob

Let $\langle S, A, \mathrm{P} \rangle$ be a generative system. A finite path π of $\langle S, A, \mathrm{P} \rangle$ is an alternating sequence $(s_0, a_1, s_1, a_2, \dots, a_k, s_k)$, where $k \in \mathbb{N}_0$, $s_i \in S$, $a_j \in A$, and $\mathrm{P}(s_{l-1}, a_l, s_l) > 0$, $l = 1, \dots, k$. We will denote a finite path $\pi = (s_0, a_1, s_1, a_2, \dots, a_k, s_k)$ more suggestively by

$$s_0 \xrightarrow{a_1} s_1 \xrightarrow{a_2} s_2 \cdots s_{k-1} \xrightarrow{a_k} s_k\,.$$

Moreover, set

$$\mathrm{length}(\pi) = k,\ \mathrm{first}(\pi) = s_0,\ \mathrm{last}(\pi) = s_k,\ \mathrm{trace}(\pi) = a_1 a_2 \cdots a_k\,.$$

The path $\epsilon = (s_0)$ will be understood as the empty path starting at s_0. Similarly, an infinite path π of $\langle S, A, \mathrm{P}\rangle$ is a sequence $(s_0, a_1, s_1, a_2, \ldots)$, where $s_i \in S$, $a_j \in A$ and $\mathrm{P}(s_{l-1}, a_l, s_l) > 0$, $l \in \mathbb{N}$, and will be written as

$$s_0 \xrightarrow{a_1} s_1 \xrightarrow{a_2} s_2 \cdots$$

Again we set $\mathrm{first}(\pi) = s_0$. A path π is called complete if it is either infinite or finite with $\mathrm{last}(\pi)$ a terminating state.

The sets of all (finite or infinite) paths, of all finite paths and of all complete paths will be denoted by Paths, FPaths and CPaths, respectively. Moreover, if $s \in S$, we write

$$\begin{aligned}
\mathrm{Paths}(s) &= \big\{\pi \in \mathrm{Paths} : \mathrm{first}(\pi) = s\big\},\\
\mathrm{FPaths}(s) &= \big\{\pi \in \mathrm{FPaths} : \mathrm{first}(\pi) = s\big\},\\
\mathrm{CPaths}(s) &= \big\{\pi \in \mathrm{CPaths} : \mathrm{first}(\pi) = s\big\}.
\end{aligned}$$

The set $\mathrm{Paths}(s)$ is partially ordered in a natural way by the prefix relation which is defined as follows. For $\pi, \pi' \in \mathrm{Paths}(s)$ we have $\pi \preceq \pi'$ if and only if one of (a), (b) or (c) holds:

(a) Both, π and π', are finite, say $\pi \equiv s \xrightarrow{a_1} s_1 \cdots \xrightarrow{a_k} s_k$, $\pi' \equiv s \xrightarrow{a_1'} s_1' \cdots \xrightarrow{a_n'} s_n'$, and we have
$$k \le n \text{ and } s_i = s_i', a_j = a_j',\ i, j \le k\,.$$

(b) π is a finite and π' an infinite path, say $\pi \equiv s \xrightarrow{a_1} s_1 \cdots \xrightarrow{a_k} s_k$, $\pi' \equiv s \xrightarrow{a_1'} s_1' \xrightarrow{a_2'} s_2' \cdots$, and we have
$$s_i = s_i', a_j = a_j',\ i, j \le k\,.$$

(c) $\pi = \pi'$

The complete paths are exactly the maximal elements in this partial order. For every $\pi \in \mathrm{Paths}(s)$, there exists a $\pi' \in \mathrm{CPaths}(s)$ such that $\pi \preceq \pi'$.

It is important to note the following:

Lemma 5.1 *For any state $s \in S$, the set* $\mathrm{FPaths}(s)$ *is at most countable.*

Proof. We first show, by induction on the length of paths, that for any fixed natural number k the number of finite paths that start in s and have length k is at most countable. For $k = 1$ the statement follows from the fact that $\mathrm{P}(s, \_, \_)$ is a probability distribution on $A \times S$ which implies that it has at most countable support set i.e. $\mathrm{P}(s, a, s') > 0$ for at most countably many pairs

$\langle a, s' \rangle \in A \times S$. Consider paths of length $n+1$. By the inductive hypothesis there are at most countably many paths of length n. Each of these can be extended to a path of length $n+1$ in at most countably many ways, hence the number of paths of length $n+1$ is also countable. Finally, the statement follows since $\mathrm{FPaths}(s) = \bigcup_{k \in \mathbb{N}_0} \{\pi \in \mathrm{FPaths}(s) : \mathrm{length}(\pi) = k\}$. $\square$

The first task is to construct out of P a probability measure on a certain σ-algebra on CPaths(s). This method was used in [BH99,Bai98], however, for the convenience of the reader we shall give complete proofs. As a standard reference for measure theoretic notions and results we use [Zaa58].

For a finite path $\pi \in \mathrm{FPaths}(s)$, let $\pi\uparrow$ denote the set

$$\pi\uparrow = \{\xi \in \mathrm{CPaths}(s) \mid \pi \preceq \xi\}$$

also called the cone of complete paths generated by the finite path π.

Note that always $\pi\uparrow \neq \emptyset$. Let

$$\Gamma := \big\{\pi\uparrow : \pi \in \mathrm{FPaths}(s)\big\} \subseteq \mathcal{P}(\mathrm{CPaths}(s))$$

denote the set of all cones. By Lemma 5.1 this set is at most countable. For the study of weak bisimulation in generative systems a thorough understanding of the geometry of cones is crucial. First of all let us state the following elementary property:

Lemma 5.2 *Let $\pi_1, \pi_2 \in \mathrm{FPaths}(s)$. Then the cones $\pi_1\uparrow$ and $\pi_2\uparrow$ are either disjoint or one is a subset of the other. In fact,*

$$\pi_1\uparrow \cap \pi_2\uparrow = \begin{cases} \pi_2\uparrow & , \ \pi_1 \preceq \pi_2 \\ \pi_1\uparrow & , \ \pi_2 \preceq \pi_1 \\ \emptyset & , \ \pi_1 \not\preceq \pi_2 \text{ and } \pi_2 \not\preceq \pi_1 \end{cases}$$

Moreover, we have $\pi_1\uparrow = \pi_2\uparrow$ if and only if either

$$\pi_1 \equiv s \xrightarrow{a_1} \cdots \xrightarrow{a_k} s_k, \ \pi_2 \equiv s \xrightarrow{a_1} \cdots \xrightarrow{a_k} s_k \xrightarrow{a_{k+1}} s_{k+1} \cdots \xrightarrow{a_n} s_n \qquad (19)$$

and thereby

$$\mathrm{P}(s_{l-1}, a_l, s_l) = 1, \ l = k+1, \dots, n \qquad (20)$$

or vice-versa.

Proof. Let $\pi \in \pi_1\uparrow \cap \pi_2\uparrow$, $\pi \in \mathrm{CPaths}(s)$. Then $\pi_1 \preceq \pi$ and $\pi_2 \preceq \pi$. This implies that $\pi_1 \preceq \pi_2$ or $\pi_2 \preceq \pi_1$. Assume $\pi_1 \preceq \pi_2$. Then

$$\pi \in \pi_2\uparrow \iff \pi_2 \preceq \pi \implies \pi_1 \preceq \pi \iff \pi \in \pi_1\uparrow .$$

It is clear that (19) and (20) imply $\pi_1\uparrow = \pi_2\uparrow$. Assume $\pi_1\uparrow = \pi_2\uparrow$. Then $\pi_1 \preceq \pi_2$ or $\pi_2 \preceq \pi_1$. Assume $\pi_1 \preceq \pi_2$, $\pi_1 \equiv s \xrightarrow{a_1} \cdots \xrightarrow{a_k} s_k$, $\pi_2 \equiv$

$s \xrightarrow{a_1} \cdots \xrightarrow{a_k} s_k \xrightarrow{a_{k+1}} s_{k+1} \cdots \xrightarrow{a_n} s_n$, and assume there exists a path $\pi_2' \neq \pi_2$,

$$\pi_2' \equiv s \xrightarrow{a_1} \cdots \xrightarrow{a_k} s_k \xrightarrow{a_{k+1}'} s_{k+1}' \cdots \xrightarrow{a_m'} s_m'.$$

Then $\pi_2' \uparrow \cap \pi_1 \uparrow = \pi_2' \uparrow$, but $\pi_2' \uparrow \cap \pi_2 \uparrow = \emptyset$ contradicting $\pi_1 \uparrow = \pi_2 \uparrow$. $\square$

Let $\Pi \subseteq \mathrm{FPaths}(s)$. We say that Π is minimal if for any two $\pi_1, \pi_2 \in \Pi$, $\pi_1 \neq \pi_2$, we have $\pi_1 \uparrow \cap \pi_2 \uparrow = \emptyset$. We will express that Π is minimal by writing $\min(\Pi)$. As example note that every singleton set $\{\pi\}$, $\pi \in \mathrm{FPaths}(s)$, is minimal.

For $\Pi \subseteq \mathrm{FPaths}(s)$ we denote by $\Pi \uparrow$ the set

$$\Pi \uparrow := \bigcup_{\pi \in \Pi} \pi \uparrow \ .$$

Then the fact $\min(\Pi)$ just means that $\Pi \uparrow$ is actually the disjoint union of all $\pi \uparrow$, $\pi \in \Pi$, i.e.

$$\min(\Pi) \text{ if and only if } \Pi \uparrow = \bigsqcup_{\pi \in \Pi} \pi \uparrow ,$$

where, here and in the sequel, the symbol $\sqcup$ denotes disjoint unions. It is an immediate consequence of the definition that, if $\min(\Pi)$ and $\Pi' \subseteq \Pi$, then also $\min(\Pi')$.

If Π_1 and Π_2 are minimal, their union need not necessarily be minimal, even if $\Pi_1 \cap \Pi_2 = \emptyset$. We will use the notation $\Pi = \biguplus_{i \in I} \Pi_i$ to express that

$$\Pi_i \subseteq \mathrm{FPaths}(s),\ i \in I,\ \Pi = \bigsqcup_{i \in I} \Pi_i \text{ and } \min(\Pi) .$$

Note that if $\Pi = \biguplus_{i \in I} \Pi_i$, also $\min(\Pi_i)$ for all $i \in I$. In particular this notation applies to minimal subsets Π written as the union of their one-element subsets:

$$\Pi = \biguplus_{\pi \in \Pi} \{\pi\} \text{ whenever } \min(\Pi).$$

Observe that the following properties hold:

(i) If $\Pi = \biguplus_{i \in I} \Pi_i$, then

$$\Pi \uparrow = \bigsqcup_{i \in I} \Pi_i \uparrow = \bigsqcup_{i \in I, \pi \in \Pi_i} \pi \uparrow \ .$$

(ii) We have $\Pi = \biguplus_{i \in I} \Pi_i$ if and only if

$$\min(\Pi_i),\ i \in I,\ \Pi_i \cap \Pi_j = \emptyset,\ i \neq j,$$

$$\pi_i \not\leq \pi_j, \pi_j \not\leq \pi_i,\ \pi_i \in \Pi_i, \pi_j \in \Pi_j,\ i \neq j .$$

Lemma 5.3 *Let* $\Pi \subseteq \mathrm{FPaths}(s)$*. Then there exists a unique set* $\Pi\downarrow \subseteq \mathrm{FPaths}(s)$*, such that*

(*i*) $\Pi\downarrow \subseteq \Pi$, $\min(\Pi\downarrow)$*, and*

$$\Pi\uparrow = \Big(\Pi\downarrow\Big)\uparrow .$$

(*ii*) *For every set* $\Pi' \subseteq \mathrm{FPaths}(s)$ *which possesses the property* (*i*)*, we have*

$$\forall \pi' \in \Pi' \; \exists \pi \in \Pi\downarrow : \; \pi \preceq \pi'$$

Proof. Take

$$\Pi\downarrow = \{\pi \in \Pi \mid \forall \pi' \in \Pi : \pi' \not\prec \pi\}.$$

If $\Pi \neq \emptyset$, then $\Pi\downarrow \neq \emptyset$ since there are no infinite prefix descending sequences. Clearly, $\min(\Pi\downarrow)$ and $\Pi\downarrow \subseteq \Pi$. Therefore, $(\Pi\downarrow)\uparrow \subseteq \Pi\uparrow$. Note that $\forall \pi \in \Pi, \exists \pi' \in \Pi\downarrow : \pi' \preceq \pi$. Hence, by Lemma 5.2, for any $\pi \in \Pi$, there exists $\pi' \in \Pi\downarrow$ such that $\pi\uparrow \subseteq \pi'\uparrow$ i.e. $\Pi\uparrow \subseteq (\Pi\downarrow)\uparrow$ and we have shown (*i*). Let Π' be a set that satisfies (*i*), i.e., $\Pi' \subseteq \Pi$, $\min(\Pi')$ and $\Pi\uparrow = \Pi'\uparrow$. Let $\pi' \in \Pi'$. Then $\pi' \in \Pi$ and as noted before there exists $\pi \in \Pi\downarrow$ such that $\pi \preceq \pi'$, proving (*ii*). The uniqueness follows from (*ii*) and the minimality of $\Pi\downarrow$. □

Lemma 5.4 *The set* $\Gamma \cup \{\emptyset\}$ *is a semi-ring (in the sense of [Zaa58]).*

Proof. Clearly, $\Gamma \cup \{\emptyset\}$ contains the empty set and it is closed under intersection, by Lemma 5.2. We need to check that the set-difference of any two of its elements is a disjoint union of at most countably many elements of $\Gamma \cup \{\emptyset\}$. Let $\pi_1\uparrow, \pi_2\uparrow \in \Gamma$. By Lemma 5.2, the only interesting case is when $\pi_1\uparrow \subset \pi_2\uparrow$, implying $\pi_2 \prec \pi_1$ (or symmetrically, $\pi_1 \prec \pi_2$). Let

$$\pi_2 \equiv s \xrightarrow{a_1} \cdots \xrightarrow{a_k} s_k, \;\; \pi_1 \equiv s \xrightarrow{a_1} \cdots \xrightarrow{a_k} s_k \xrightarrow{a_{k+1}} s_{k+1} \cdots \xrightarrow{a_n} s_n, \;\; k < n$$

and put

$$\Pi = \{\pi \mid \pi \equiv s \xrightarrow{a_1} \cdots \xrightarrow{a_m} s_m \xrightarrow{b} t, \;\; k \le m < n, \;\; s \xrightarrow{a_1} \cdots \xrightarrow{a_m} s_m \prec \pi_1, \pi \not\prec \pi_1\}.$$

It is not difficult to see that $(\pi_2\uparrow) \setminus (\pi_1\uparrow) = \Pi\uparrow = \cup_{\pi \in \Pi}\, \pi\uparrow$ and the union is at most countable. □

Now we are ready to introduce the desired extension of P to a measure. By Lemma 5.2 a function $\mathrm{Prob} : \Gamma \cup \{\emptyset\} \to [0,1]$ is well defined by

$$\mathrm{Prob}(E) := \begin{cases} \mathrm{P}(s, a_1, s_1) \cdot \ldots \cdot \mathrm{P}(s_{k-1}, a_k, s_k) & , \begin{array}{l} E = \pi\uparrow \text{ with } k \ge 1, \\ \pi = s \xrightarrow{a_1} s_1 \cdots s_{k-1} \xrightarrow{a_k} s_k \end{array} \\ 1 & , E = \epsilon\uparrow = \mathrm{CPaths}(s) \\ 0 & , E = \emptyset \end{cases}$$

Lemma 5.5 *The function* Prob *is a measure on the semi-ring* $\Gamma \cup \{\emptyset\}$.

Proof. By definition $\mathrm{Prob}(\emptyset) = 0$. We need to check σ-aditivity and monotonicity. Assume $\pi \uparrow = \sqcup_{i \in I} \pi_i \uparrow$ for some at most countable index set I. This is only possible if π is not complete and if $\{\pi_i \mid i \in I\} = \{\pi' \xrightarrow{a} s \mid a \in A, s \in S\}$ for $\pi' \uparrow = \pi \uparrow$ (see Lemma 5.2), i.e. π_i for $i \in I$ are exactly the paths that extend π' in one step, for π' a trivial extension of π. Such paths exist at most countably many. Then

$$\begin{aligned}\sum_{i=1}^{\infty} \mathrm{Prob}(\pi_i \uparrow) &= \mathrm{Prob}(\pi' \uparrow) \cdot \sum_{a \in A, s \in S} \mathrm{P}(\mathrm{last}(\pi'), a, s) \\ &\overset{(*)}{=} \mathrm{Prob}(\pi' \uparrow) \\ &= \mathrm{Prob}(\pi \uparrow)\end{aligned}$$

where $(*)$ holds since π' does not end in a terminating state, i.e.

$$\sum_{a \in A, s \in S} \mathrm{P}(\mathrm{last}(\pi), a, s) = 1.$$

The function Prob is monotonic by definition: Assume $\pi_1 \uparrow \subseteq \pi_2 \uparrow$. Then, by Lemma 5.2, $\pi_2 \preceq \pi_1$ and since $\mathrm{P}(s, a, t) \leq 1$ for all $s, t \in S, a \in A$, from the definition of Prob we get $\mathrm{Prob}(\pi_1 \uparrow) \leq \mathrm{Prob}(\pi_2 \uparrow)$. □

Corollary 5.6 *The function* Prob *extends uniquely to a probability measure on the* σ*-algebra on* $\mathrm{CPaths}(s)$ *generated by* $\Gamma \cup \{\emptyset\}$. *We will denote this measure again by* Prob.

Remark 5.7 Note that, although paths are more or less just sequences of elements of S and A, not only the function Prob itself, but also the σ-algebra where it is defined and in fact already the base set $\mathrm{CPaths}(s)$ depends heavily on P. At the first sight this might seem to be an undesirable fact, however, a second look at the matters shows that it cannot be avoided.

The measure Prob induces a set-function on finite paths, which we will also denote by Prob. Define $\mathrm{Prob} : \mathcal{P}(\mathrm{FPaths}(s)) \to [0, 1]$ by

$$\mathrm{Prob}(\Pi) = \mathrm{Prob}(\Pi \uparrow).$$

This notation is not in conflict with the already existing notation of the measure Prob. In fact, $\mathcal{P}(\mathrm{FPaths}(s)) \cap \mathcal{P}(\mathrm{CPaths}(s))$ consists entirely of Prob-measureable sets and on such sets both definitions coincide. To see this, note that if $\pi \in \mathrm{FPaths}(s) \cap \mathrm{CPaths}(s)$, then $\pi \uparrow = \{\pi\}$. Thus, if $\Pi \in \mathcal{P}(\mathrm{FPaths}(s)) \cap \mathcal{P}(\mathrm{CPaths}(s))$, we have

$$\Pi = \bigsqcup_{\pi \in \Pi} \{\pi\} = \bigsqcup_{\pi \in \Pi} \pi \uparrow = \Pi \uparrow,$$

and this union is at most countable.

It will always be clear from the context whether we mean the measure Prob or the just defined set-function Prob. Still, there is a word of caution in order: The function $\mathrm{Prob} : \mathcal{P}(\mathrm{FPaths}(s)) \to [0,1]$ is in general not additive. However, having a look at the notations introduced above, we find that

$$\mathrm{Prob}(\Pi) = \sum_{i \in I} \mathrm{Prob}(\Pi_i), \text{ whenever } \Pi = \biguplus_{i \in I} \Pi_i \,.$$

In particular, we obtain that $\mathrm{Prob}(\Pi) = \sum_{\pi \in \Pi} \mathrm{Prob}(\pi \uparrow)$ for every minimal set Π. Moreover, by Lemma 5.3, we always have

$$\mathrm{Prob}(\Pi) = \mathrm{Prob}(\Pi \downarrow) \,.$$

We next introduce some particular sets of paths. For $s \in S$, $S', S'' \subseteq S$ with $S' \subseteq S''$, and $W, W' \subseteq A^*$ with $W \subseteq W'$, denote

$$s \overset{W}{\underset{\neg S''}{\rightarrow_{\neg W'}}} S' = \left\{ \pi \in \mathrm{FPaths}(s) \mid \begin{array}{c} \mathrm{last}(\pi) \in S', \mathrm{trace}(\pi) \in W \\ \forall\, \xi \prec \pi : \mathrm{trace}(\xi) \in W' \Rightarrow \mathrm{last}(\xi) \notin S'' \end{array} \right\}$$

and write $\mathrm{Prob}(s, W, \neg W, S', \neg S'') = \mathrm{Prob}(s \overset{W}{\underset{\neg S''}{\rightarrow_{\neg W'}}} S')$. Since $S' \subseteq S''$ and $W \subseteq W'$ we always have $\min(s \overset{W}{\underset{\neg S''}{\rightarrow_{\neg W'}}} S')$. For notational convenience we will drop redundant arguments whenever possible. Put

$$\begin{aligned} s \overset{W}{\rightarrow}_{\neg W'} S' &= s \overset{W}{\underset{\neg S'}{\rightarrow_{\neg W'}}} S', \\ s \overset{W}{\rightarrow}_{\neg S''} S' &= s \overset{W}{\underset{\neg S''}{\rightarrow_{\neg W}}} S', \\ s \overset{W}{\rightarrow} S' &= s \overset{W}{\underset{\neg S'}{\rightarrow_{\neg W}}} S' \,, \end{aligned} \tag{21}$$

and, correspondingly,

$$\begin{aligned} \mathrm{Prob}(s, W, \neg W', S') &= \mathrm{Prob}(s, W, \neg W', S', \neg S'), \\ \mathrm{Prob}(s, W, S', \neg S'') &= \mathrm{Prob}(s, W, \neg W, S', \neg S''), \\ \mathrm{Prob}(s, W, S') &= \mathrm{Prob}(s, W, \neg W, S', \neg S') \,. \end{aligned} \tag{22}$$

Note that

$$s \overset{W}{\rightarrow} S' = \left\{ \pi \in \mathrm{FPaths}(s) : \ \mathrm{last}(\pi) \in S', \mathrm{trace}(\pi) \in W \right\} \downarrow \,.$$

Let S', S'', W, W' be as above and let moreover $F \subseteq S$ be given. Then denote

$$F \overset{W}{\underset{\neg S''}{\rightarrow_{\neg W'}}} S' = \bigsqcup_{s \in F} s \overset{W}{\underset{\neg S''}{\rightarrow_{\neg W'}}} S' \subseteq \mathrm{FPaths}$$

We will often encounter the situation that for every $s \in F$ the value of $\mathrm{Prob}(s, W, \neg W', S', \neg S'')$ is the same. In this case we speak of this value as

$\mathrm{Prob}(F, W, \neg W', S', \neg S'')$. Also, in this context, we shall freely apply shortening of notation as in (21) and (22).
Next we define sets of concatenated paths. For $\Pi \subseteq \mathrm{FPaths}$, put

$$\mathrm{first}(\Pi) = \{\mathrm{first}(\pi) \mid \pi \in \Pi\},\ \mathrm{last}(\Pi) = \{\mathrm{last}(\pi) \mid \pi \in \Pi\}.$$

If $\Pi_1, \Pi_2 \subseteq \mathrm{FPaths}$ and $\mathrm{last}(\Pi_1) = \mathrm{first}(\Pi_2)$, we define

$$\Pi_1 \cdot \Pi_2 = \big\{\pi_1 \cdot \pi_2 \mid \pi_1 \in \Pi_1, \pi_2 \in \Pi_2, \mathrm{last}(\pi_1) = \mathrm{first}(\pi_2)\big\},$$

where $\pi_1 \cdot \pi_2 \equiv s \stackrel{a_1}{\to} \cdots \stackrel{a_k}{\to} s_k \stackrel{a_{k+1}}{\to} \cdots \stackrel{a_n}{\to} s_n$ for $\pi_1 \equiv s \stackrel{a_1}{\to} \cdots \stackrel{a_k}{\to} s_k$ and $\pi_2 \equiv s_k \stackrel{a_{k+1}}{\to} \cdots \stackrel{a_n}{\to} s_n$. Note that, whenever a concatenation $\pi_1 \cdot \pi_2$ is defined, we have $\mathrm{Prob}(\{\pi_1 \cdot \pi_2\}) = \mathrm{Prob}(\{\pi_1\}) \cdot \mathrm{Prob}(\{\pi_2\})$.

Proposition 5.8 *Let* $\Pi_1 \subseteq \mathrm{FPaths}(s)$, $\Pi_2 \subseteq \mathrm{FPaths}$ *with* $\mathrm{last}(\Pi_1) = \mathrm{first}(\Pi_2)$ *and assume that this set is represented as a disjoint union*

$$\mathrm{last}(\Pi_1) = \mathrm{first}(\Pi_2) = \bigsqcup_{i \in I} S_i.$$

Denote $\Pi_{1,S_i} = \{\pi_1 \in \Pi_1 : \mathrm{last}(\pi_1) \in S_i\}$, $\Pi_{2,t} = \{\pi_2 \in \Pi_2 : \mathrm{first}(\pi_2) = t\}$. *Assume that for every* $i \in I$

$$\mathrm{Prob}(\Pi_{2,t'}) = \mathrm{Prob}(\Pi_{2,t''}),\ t', t'' \in S_i.$$

Moreover, assume that Π_1, Π_2 *and* $\Pi_1 \cdot \Pi_2$ *are minimal. Then, for every choice of* $(t_i)_{i \in I} \in \prod_{i \in I} S_i$, *we have*

$$\mathrm{Prob}(\Pi_1 \cdot \Pi_2) = \sum_{i \in I} \mathrm{Prob}(\Pi_{1,S_i}) \cdot \mathrm{Prob}(\Pi_{2,t_i}).$$

Proof. Denote by $\Pi_{2,S_i} = \{\pi_2 \in \Pi_2 \mid \mathrm{first}(\pi_2) \in S_i\}$ and by $\Pi_{1,t} = \{\pi_1 \in \Pi_1 \mid \mathrm{last}(\pi_1) = t\}$. Under the assumptions of the proposition, we have

$$\begin{aligned}
\mathrm{Prob}(\Pi_1 \cdot \Pi_2) &= \mathrm{Prob}(\biguplus_{\pi \in \Pi_1 \cdot \Pi_2} \pi \uparrow) \\
&= \mathrm{Prob}(\biguplus_{i \in I}(\biguplus_{\pi \in \Pi_{1,S_i} \cdot \Pi_{2,S_i}} \pi \uparrow)) \\
&= \mathrm{Prob}(\biguplus_{i \in I}(\biguplus_{t \in S_i}(\biguplus_{\pi \in \Pi_{1,t} \cdot \Pi_{2,t}} \pi \uparrow))) \\
&= \sum_{i \in I} \sum_{t \in S_i} \sum_{\pi \in \Pi_{1,t} \cdot \Pi_{2,t}} \mathrm{Prob}(\pi \uparrow)
\end{aligned}$$

Since $\Pi_{1,t} \times \Pi_{2,t} \cong \Pi_{1,t} \cdot \Pi_{2,t}$ via $(\pi_1, \pi_2) \mapsto \pi_1 \cdot \pi_2$, we have

$$\begin{aligned}
\sum_{\pi \in \Pi_{1,t} \cdot \Pi_{2,t}} \mathrm{Prob}(\pi \uparrow) &= \sum_{(\pi_1,\pi_2) \in \Pi_{1,t} \times \Pi_{2,t}} \mathrm{Prob}(\pi_1 \cdot \pi_2 \uparrow) \\
&= \sum_{\pi_1 \in \Pi_{1,t}} \sum_{\pi_2 \in \Pi_{2,t}} \mathrm{Prob}(\pi_1 \uparrow)\, \mathrm{Prob}(\pi_2 \uparrow) \\
&= \sum_{\pi_1 \in \Pi_{1,t}} \mathrm{Prob}(\pi_1 \uparrow) \cdot \sum_{\pi_2 \in \Pi_{2,t}} \mathrm{Prob}(\pi_2 \uparrow) \\
&= \mathrm{Prob}(\Pi_{1,t}) \cdot \mathrm{Prob}(\Pi_{2,t})\,.
\end{aligned}$$

Since for every $i \in I$ the value of $\mathrm{Prob}(\Pi_{2,t})$ does not depend on $t \in S_i$, it follows that

$$\begin{aligned}
\mathrm{Prob}(\Pi_1 \cdot \Pi_2) &= \sum_{i \in I} \Big(\mathrm{Prob}(\Pi_{2,t_i}) \cdot \sum_{t \in S_i} \mathrm{Prob}(\Pi_{1,t}) \Big) \\
&= \sum_{i \in I} \mathrm{Prob}(\Pi_{2,t_i})\, \mathrm{Prob}(\Pi_{1,S_i})\,.
\end{aligned}$$

□

It is worth to explicitly note the particular case of this proposition when $|I| = 1$.

Corollary 5.9 *Let* $\Pi_1 \subseteq \mathrm{FPaths}(s)$, $\Pi_2 \subseteq \mathrm{FPaths}$ *with* $\mathrm{last}(\Pi_1) = \mathrm{first}(\Pi_2)$. *Let* $\Pi_{2,t} = \{\pi_2 \in \Pi_2 \mid \mathrm{first}(\pi_2) = t\}$. *Then, if* $\min(\Pi_1)$, $\min(\Pi_2)$ *and* $\min(\Pi_1 \cdot \Pi_2)$, *and if for any* $t', t'' \in \mathrm{first}(\Pi_2)$, $\mathrm{Prob}(\Pi_{2,t'}) = \mathrm{Prob}(\Pi_{2,t''})$, *we have that*

$$\mathrm{Prob}(\Pi_1 \cdot \Pi_2) = \mathrm{Prob}(\Pi_1) \cdot \mathrm{Prob}(\Pi_{2,t})$$

for arbitrary $t \in \mathrm{first}(\Pi_2)$. □

5.2 Weak coalgebraic bisimulation for generative systems

For treating weak probabilistic bisimulation, we shall need to consider one more type of systems. Let $\mathcal{G}^*$ be the bifunctor defined by

$$\mathcal{G}^*(A, S) = (\mathcal{P}(A) \times \mathcal{P}(S) \to [0,1])$$

on objects $\langle A, S\rangle$ and for morphisms $\langle f_1, f_2\rangle \colon A \times S \to B \times T$ by

$$\mathcal{G}^* f = (\nu \mapsto \nu \circ \langle f_1^{-1}, f_2^{-1}\rangle \mid \nu \colon \mathcal{P}(A) \times \mathcal{P}(S) \to [0,1]).$$

Consider the Set functor $\mathcal{G}^*_A$ corresponding to $\mathcal{G}^*$, so that $\mathcal{G}^*_A(S) = (\mathcal{P}(A) \times \mathcal{P}(S) \to [0,1])$ and for a mapping $f \colon S \to T$, $\mathcal{G}^*_A f = (\nu \mapsto \nu \circ \langle id_A^{-1}, f^{-1}\rangle \mid \nu \colon \mathcal{P}(A) \times \mathcal{P}(S) \to [0,1])$. We will use the functor $\mathcal{G}^*_A$ to model the $*$-translation of generative systems. Therefore we are interested in characterizing equivalence bisimulations for this functor. In order to apply Lemma 2.11 we need the following.

Lemma 5.10 *The functor* $\mathcal{G}^*_A$ *weakly preserves total pullbacks.*

Proof. Let $\langle P, \pi_1, \pi_2\rangle$ be a total Set pullback of the cospan $X \xrightarrow{f} Z \xleftarrow{g} Y$ i.e. $P = \{\langle x, y\rangle \mid f(x) = g(y)\}$ and π_1, π_2 surjective. Then the outer square of the following diagram commutes, and a morphism $\gamma : \mathcal{G}^*_A P \to P'$ exists, where P' is the Set pullback of the cospan $\mathcal{G}^*_A X \xrightarrow{\mathcal{G}^*_A f} \mathcal{G}^*_A Z \xleftarrow{\mathcal{G}^*_A g} \mathcal{G}^*_A Y$.

$$
\begin{tikzcd}
& \mathcal{G}^*_A P \arrow[d, dashed, "\gamma"] \arrow[ddl, "\mathcal{G}^*_A\pi_1"'] \arrow[ddr, "\mathcal{G}^*_A\pi_2"] & \\
& P' \arrow[dl, "\pi_1"] \arrow[dr, "\pi_2"'] & \\
\mathcal{G}^*_A X \arrow[dr, "\mathcal{G}^*_A f"] & & \mathcal{G}^*_A Y \arrow[dl, "\mathcal{G}^*_A g"'] \\
& \mathcal{G}^*_A Z &
\end{tikzcd}
$$

It is enough to prove that γ is surjective. Since π_1 and π_2 are jointly injective, this is to show that for every $\langle u, v\rangle \in P'$ there exists $w \in \mathcal{G}^*_A P$ with $w \circ \langle id_A^{-1}, \pi_1^{-1}\rangle = u$ and $w \circ \langle id_A^{-1}, \pi_2^{-1}\rangle = v$. Fix $\langle u, v\rangle \in P'$. Note the following

(a) $\langle u, v\rangle \in P'$ implies that $\forall A' \subseteq A, \forall Z' \subseteq Z\colon u(A', f^{-1}(Z')) = v(A', g^{-1}(Z'))$.

(b) $\pi_1^{-1}(X') = \pi_1^{-1}(X'') \implies X' = X''$ for any $X', X'' \subseteq X$, since π_1 is surjective.

(c) $\pi_2^{-1}(Y') = \pi_2^{-1}(Y'') \implies Y' = Y''$ for any $Y', Y'' \subseteq Y$, since π_2 is surjective.

(d) Let $X' \subseteq X, Y' \subseteq Y$. Then $\pi_1^{-1}(X') = \pi_2^{-1}(Y')$ implies

(d1) $f^{-1}(f(X')) = X'$
Clearly $X' \subseteq f^{-1}(f(X'))$. Let $x' \in f^{-1}(f(X'))$ such that $f(x') = f(x)$ for some $x \in X'$. Since π_1 is surjective, there exists $y \in Y$ with $\langle x', y\rangle \in P$ i.e. $f(x') = g(y)$ and hence also $f(x) = g(y)$ i.e. $\langle x, y\rangle \in P$. Thus $\langle x, y\rangle \in \pi_1^{-1}(X') = \pi_2^{-1}(Y')$ from where $y \in Y'$. Hence $\langle x', y\rangle \in \pi_2^{-1}(Y') = \pi_1^{-1}(X')$ i.e. $x' \in X'$.

(d2) $g^{-1}(g(Y')) = Y'$, similar as (d1).

(d3) $f(X') = g(Y')$
Let $z \in f(X')$ i.e. $z = f(x)$ for $x \in X'$. Since π_1 is surjective there exists $y \in Y$ with $\langle x, y\rangle \in P$ i.e. $f(x) = g(y)$. Now $\langle x, y\rangle \in \pi_1^{-1}(X') = \pi_2^{-1}(Y')$ and therefore $y \in Y'$ i.e. $z = f(x) = g(y) \in g(Y')$. Hence $f(X') \subseteq g(Y')$. Similarly, $g(Y') \subseteq f(X')$.

Hence, if $\pi_1^{-1}(X') = \pi_2^{-1}(Y')$ for $X' \subseteq X, Y' \subseteq Y$ we get, for any $A' \subseteq A$

$$u(A', X') \stackrel{(d1)}{=} u(A', f^{-1}(f(X'))) \stackrel{(a)}{=} v(A', g^{-1}(f(X'))) \stackrel{(d3)}{=}$$

$$v(A', g^{-1}(g(Y'))) \stackrel{(d2)}{=} v(A', Y').$$

This, together with (b) and (c) shows that the function $w\colon \mathcal{P}(A) \times \mathcal{P}(P) \to$

$[0,1]$ given by

$$w(A', Q) = \begin{cases} u(A', X') & Q = \pi_1^{-1}(X') \\ v(A', Y') & Q = \pi_2^{-1}(Y') \\ 0 & \text{otherwise} \end{cases}$$

is well defined. Clearly, $w \circ \langle id_A^{-1}, \pi_1^{-1} \rangle = u$ and $w \circ \langle id_A^{-1}, \pi_2^{-1} \rangle = v$. Thus γ is surjective. $\square$

Note that, however, $\mathcal{G}_A^*$ does not preserve weak pullbacks, as shown by the next example.

Example 5.11 $\mathcal{G}_A^*$ does not preserve weak pullbacks.

Choose X with $|X| \geq 3$. Fix $x_0 \in X$. Let $Z = \{1, 2, 3\}$ and consider the cospan $X \xrightarrow{f} Z \xleftarrow{g} X$ for the maps

$$f(x) = \begin{cases} 2 & x = x_0 \\ 1 & \text{otherwise} \end{cases} \qquad g(x) = \begin{cases} 2 & x = x_0 \\ 3 & \text{otherwise.} \end{cases}$$

The Set pullback of this cospan is then $P = \{\langle x_0, x_0 \rangle\}$. On the other hand, let P' be the pullback of the cospan

$$\mathcal{G}_A^* X \xrightarrow{\mathcal{G}_A^* f} \mathcal{G}_A^* Z \xleftarrow{\mathcal{G}_A^* g} \mathcal{G}_A^* X \, .$$

Then every pair $\langle \mu, \nu \rangle \in \mathcal{G}_A^* X \times \mathcal{G}_A^* X$ with the property

$$\mu(A', \emptyset) = \mu(A', \{x_0\}) = \mu(A', X \setminus \{x_0\}) = \mu(A', X) =$$

$$= \nu(A', \emptyset) = \nu(A', \{x_0\}) = \nu(A', X \setminus \{x_0\}) = \nu(A', X)$$

belongs to P' since $\emptyset, \{x_0\}, X \setminus \{x_0\}$ and X are the only subsets of X that are inverse images of subsets of Z under f and g. Now we consider $\mathcal{G}_A^* P = \{\mu \colon \mathcal{P}(A) \times \mathcal{P}(\{\langle x_0, x_0 \rangle\}) \to [0,1]\}$. If $\nu \in \mathcal{G}_A^* X$ is such that $\nu = (\mathcal{G}_A^* \pi_1)(\mu)$ for some $\mu \in \mathcal{G}_A^* P$ then $\nu = \mu \circ \langle id_A^{-1}, \pi_1^{-1} \rangle$. Hence for $A' \subseteq A, X' \subseteq X$ we have

$\nu(A', X') = \mu(A', \emptyset)$ if $x_0 \notin X'$ and

$\nu(A', X') = \mu(A', \{\langle x_0, x_0 \rangle\})$ if $x_0 \in X'$.

Choose $x_1 \in X$, $x_1 \neq x_0$. Since $|X| \geq 3$ we have $\{x_0, x_1\} \notin \{\emptyset, \{x_0\}, X \setminus \{x_0\}, X\}$. Define $\xi \colon \mathcal{P}(A) \times \mathcal{P}(X) \to [0,1]$ by

$$\xi(A', X') = \begin{cases} 1 & X' = \{x_0, x_1\} \\ 0 & \text{otherwise} \end{cases}$$

Then $\langle \xi, \xi \rangle \in P'$ since

$$\xi(A', \emptyset) = \xi(A', \{x_0\}) = \xi(A', X \setminus \{x_0\}) = \xi(A', X) = 0.$$

But ξ can not be written as $(\mathcal{G}_A^* \pi_1)(\mu)$ for any $\mu \in \mathcal{G}_A^* P$ since

$$\xi(A', \{x_0, x_1\}) \neq \xi(A', \{x_0\})$$

while, as noted above,

$$(\mathcal{G}_A^* \pi_1)(\mu)(A', \{x_0, x_1\}) = \mu(A', \{\langle x_0, x_0 \rangle\}) = (\mathcal{G}_A^* \pi_1)(\mu)(A', \{x_0\}).$$

Hence there can not exist a map γ making the following diagram commute

$$\begin{tikzcd} & P' \arrow[dl, "\pi_1"'] \arrow[d, "\gamma"] \\ \mathcal{G}_A^* X & \mathcal{G}_A^* P \arrow[l, "\mathcal{G}_A^* \pi_1"'] \end{tikzcd}$$

and hence $\mathcal{G}_A^* P$ can not be a weak pullback of $\mathcal{G}_A^* X \xrightarrow{\mathcal{G}_A^* f} \mathcal{G}_A^* Z \xleftarrow{\mathcal{G}_A^* g} \mathcal{G}_A^* X$.

Let R be an equivalence relation on a set S. A subset $M \subseteq S$ is an R-saturated set if for all $s \in M$ the whole equivalence class of s is contained in M. We denote by $\mathrm{Sat}(R)$ the set of all R-saturated sets, $\mathrm{Sat}(R) \subseteq \mathcal{P}(S)$. Actually, M is a saturated set if and only if $M = \cup_{i \in I} C_i$ for $C_i \in S/R$. Hence there is a one-to-one correspondence between the R-saturated sets and the elements of $\mathcal{P}(S/R)$.

Lemma 5.12 *An equivalence relation R on a set S is a bisimulation on the $\mathcal{G}_A^*$ system $\langle S, A, \alpha \rangle$ according to Definition 2.3 for the functor $\mathcal{G}_A^*$ if and only if*

$$\langle s, t \rangle \in R \implies \forall A' \subseteq A, \forall M \in \mathrm{Sat}(R)\colon \alpha(s)(A', M) = \alpha(t)(A', M).$$

Proof. Consider the pullback P of the cospan $\mathcal{G}_A^* S \xrightarrow{\mathcal{G}_A^* c} \mathcal{G}_A^*(S/R) \xleftarrow{\mathcal{G}_A^* c} \mathcal{G}_A^* S$, where c is the canonical projection of S onto S/R. We have $\langle \mu, \nu \rangle \in P$ if and only if $\mathcal{G}_A^* c(\mu) = \mathcal{G}_A^* c(\nu)$ i.e. $\mu \circ \langle id_A^{-1}, c^{-1} \rangle = \nu \circ \langle id_A^{-1}, c^{-1} \rangle$ which is equivalent to

$$\forall A' \subseteq A, \forall M \subseteq S/R\colon \mu(A', c^{-1}(M)) = \nu(A', c^{-1}(M))$$

i.e., since $c^{-1} : \mathcal{P}(S/R) \to \mathrm{Sat}(R)$ is a bijection,

$$\forall A' \subseteq A, \forall M \in \mathrm{Sat}(R)\colon \mu(A', M) = \nu(A', M).$$

Now using Lemma 2.10 and Lemma 2.11 we obtain the stated characterization. □

We proceed by presenting the $*$-translation for generative systems.

Definition 5.13 Let Φ^g assign to every generative system $\langle S, A, P\rangle$ i.e. any $\mathcal{G}_A$ coalgebra $\langle S, A, \alpha\rangle$ the $\mathcal{G}^*_{A^*}$ coalgebra $\langle S, A^*, \alpha'\rangle$ where for $W \subseteq A^*$ and $S' \subseteq S$, $\alpha'(s)(W, S') = \mathrm{Prob}(s, W, S')$.

Theorem 5.14 *The assignment Φ^g from Definition 5.13 is a $*$-translation.*

For the proof we need an auxiliary property.

Lemma 5.15 *Let $\langle S, A, \alpha\rangle$, i.e. $\langle S, A, \mathrm{P}\rangle$ be a $\mathcal{G}_A$ system, R a bisimulation equivalence on $\langle S, A, \alpha\rangle$ and $\langle s, t\rangle \in R$. For $k \in \mathbb{N}, C_i \in S/R$ and $a_i \in A$, let $s \stackrel{a_1}{\to} C_1 \stackrel{a_2}{\to} C_2 \cdots \stackrel{a_k}{\to} C_k$ denote the set of paths*

$$s \stackrel{a_1}{\to} C_1 \stackrel{a_2}{\to} C_2 \cdots \stackrel{a_k}{\to} C_k = \{s \stackrel{a_1}{\to} s_1 \stackrel{a_2}{\to} s_2 \cdots \stackrel{a_k}{\to} s_k \mid s_i \in C_i, i = 1, \dots, k\}.$$

Then $s \stackrel{a_1}{\to} C_1 \stackrel{a_2}{\to} C_2 \cdots \stackrel{a_k}{\to} C_k$ is minimal and

$$\mathrm{Prob}(s \stackrel{a_1}{\to} C_1 \stackrel{a_2}{\to} C_2 \cdots \stackrel{a_k}{\to} C_k) = \mathrm{Prob}(t \stackrel{a_1}{\to} C_1 \stackrel{a_2}{\to} C_2 \cdots \stackrel{a_k}{\to} C_k) \qquad (23)$$

Proof. The fact that $s \stackrel{a_1}{\to} C_1 \stackrel{a_2}{\to} C_2 \cdots \stackrel{a_k}{\to} C_k$ is minimal is clear, since all paths in this set have the same length. We use induction on k to establish (23). For $k = 1$ the statement is $\sum_{s' \in C_1} \mathrm{P}(s, a_1, s') = \sum_{s' \in C_1} \mathrm{P}(t, a_1, s')$ and it holds since R is a bisimulation relation and $\langle s, t\rangle \in R$. Consider

$$s \stackrel{a_1}{\to} C_1 \stackrel{a_2}{\to} C_2 \cdots \stackrel{a_{k+1}}{\to} C_{k+1} = s \stackrel{a_1}{\to} C_1 \stackrel{a_2}{\to} C_2 \cdots \stackrel{a_k}{\to} C_k \cdot C_k \stackrel{a_{k+1}}{\to} C_{k+1}.$$

By the inductive hypothesis,

$$\mathrm{Prob}(s \stackrel{a_1}{\to} C_1 \stackrel{a_2}{\to} C_2 \cdots \stackrel{a_k}{\to} C_k) = \mathrm{Prob}(t \stackrel{a_1}{\to} C_1 \stackrel{a_2}{\to} C_2 \cdots \stackrel{a_k}{\to} C_k).$$

By the bisimulation condition for generative systems, $\mathrm{Prob}(t' \stackrel{a_{k+1}}{\to} C_{k+1}) = \mathrm{Prob}(t'' \stackrel{a_{k+1}}{\to} C_{k+1})$ for all $t', t'' \in C_k$. Hence, by Corollary 5.9 we get

$$\begin{aligned}
&\mathrm{Prob}(s \stackrel{a_1}{\to} C_1 \stackrel{a_2}{\to} C_2 \cdots \stackrel{a_k}{\to} C_k \cdot C_k \stackrel{a_{k+1}}{\to} C_{k+1})\\
=&\mathrm{Prob}(s \stackrel{a_1}{\to} C_1 \stackrel{a_2}{\to} C_2 \cdots \stackrel{a_k}{\to} C_k) \cdot \mathrm{Prob}(C_k \stackrel{a_{k+1}}{\to} C_{k+1})\\
=&\mathrm{Prob}(t \stackrel{a_1}{\to} C_1 \stackrel{a_2}{\to} C_2 \cdots \stackrel{a_k}{\to} C_k) \cdot \mathrm{Prob}(C_k \stackrel{a_{k+1}}{\to} C_{k+1})\\
=&\mathrm{Prob}(t \stackrel{a_1}{\to} C_1 \stackrel{a_2}{\to} C_2 \cdots \stackrel{a_k}{\to} C_k \cdot C_k \stackrel{a_{k+1}}{\to} C_{k+1}).
\end{aligned}$$

$\square$

We are now prepared for the proof of Theorem 5.14.

Proof. [of Theorem 5.14] We need to check that Φ^g is injective and preserves and reflects bisimilarity. Assume $\Phi^g(\langle S, A, \alpha\rangle) = \Phi^g(\langle S, A, \beta\rangle) = \langle S, A^*, \alpha'\rangle$. Then by the definition of Prob we get that for any $s, t \in S$ and any $a \in A$, $\alpha(s)(\langle a, t\rangle) = \mathrm{P}(s, a, t) = \mathrm{Prob}(s, \{a\}, \{t\}) = \alpha'(s)(\{a\}, \{t\}) = \beta(s)(\langle a, t\rangle)$.

Reflection of bisimilarity is direct from Lemma 5.12: Assume $s \sim t$ in $\Phi^g(\langle S, A, \alpha\rangle) = \langle S, A^*, \alpha'\rangle$. Then there is an equivalence bisimulation R on

$\langle S, A^*, \alpha' \rangle$ such that $\langle s, t\rangle \in R$. By Lemma 5.12, we get that for all $W \subseteq A^*$ and for all $M \in \mathrm{Sat}(R)$ it holds that

$$\alpha'(s)(W, M) = \alpha'(t)(W, M). \tag{24}$$

In particular, for all $a \in A$ and all $C \in S/R$ we have

$$\alpha'(s)(\{a\}, C) = \alpha'(t)(\{a\}, C). \tag{25}$$

By the definition of α' we have

$$\alpha'(s)(\{a\}, C) = \mathrm{Prob}(s, \{a\}, C) = \sum_{s' \in C} \mathrm{P}(s, a, s') = \sum_{s' \in C} \alpha(s)(\langle a, s'\rangle)$$

and therefore for all $a \in A$ and all $C \in S/R$

$$\sum_{s' \in C} \alpha(s)(\langle a, s'\rangle) = \sum_{s' \in C} \alpha(t)(\langle a, s'\rangle) \tag{26}$$

which by Definition 2.8 yields that R is a bisimulation equivalence on the generative system $\langle S, A, \alpha\rangle$ i.e. $s \sim t$ in the original system.

The proof of preservation of bisimilarity uses Lemma 5.15. Let $s \sim t$ in the generative system $\langle S, A, \alpha\rangle$. Then there exists an equivalence bisimulation R with $\langle s, t\rangle \in R$. The relation R induces an equivalence R_P on $\mathrm{FPaths}(s)$ defined by

$$\langle s \xrightarrow{a_1} s_1 \xrightarrow{a_2} s_2 \cdots \xrightarrow{a_k} s_k \ , \ s \xrightarrow{a'_1} s'_1 \xrightarrow{a'_2} s'_2 \cdots \xrightarrow{a'_{k'}} s'_{k'} \rangle \in R_P$$

if and only if $k = k'$, $a_i = a'_i$ and $\langle s_i, s'_i\rangle \in R$ for $i = 1, \ldots, k$. The classes of R_P are exactly the sets $s \xrightarrow{a_1} C_1 \xrightarrow{a_2} C_2 \cdots \xrightarrow{a_k} C_k$ for $C_i \in S/R$ and $a_i \in A$.

Assume $M \in \mathrm{Sat}(R)$ and $W \subseteq A^*$. We show that the set $s \xrightarrow{W} M$ is saturated with respect to R_P. Namely, let $\pi \equiv s \xrightarrow{a_1} s_1 \xrightarrow{a_2} s_2 \cdots \xrightarrow{a_k} s_k \in \ s \xrightarrow{W} M$ and let $\pi' \equiv s \xrightarrow{a_1} s'_1 \xrightarrow{a_2} s'_2 \cdots \xrightarrow{a_k} s'_k$ be a path such that $\langle \pi, \pi'\rangle \in R_P$. Then $\mathrm{trace}(\pi) = \mathrm{trace}(\pi')$, $\mathrm{first}(\pi) = \mathrm{first}(\pi')$ and $\langle \mathrm{last}(\pi), \mathrm{last}(\pi')\rangle \in R$. Since M is saturated, $\mathrm{last}(\pi') \in M$. Furthermore, π' does not have a proper prefix with trace in W and last in M, since this would imply that π has such a prefix, contradicting $\pi \in s \xrightarrow{W} M$. Hence, $\pi' \in s \xrightarrow{W} M$.

Therefore, the set $s \xrightarrow{W} M$ is a disjoint union of some R_P classes, and since $s \xrightarrow{W} M$ is minimal we can write

$$s \xrightarrow{W} M = \biguplus_{i \in I} s \xrightarrow{a_{i1}} C_{i1} \xrightarrow{a_{i2}} C_{i2} \cdots \xrightarrow{a_{ik_i}} C_{ik_i} \, .$$

It follows that $\mathrm{Prob}(s, W, M) = \sum_{i \in I} \mathrm{Prob}(s \xrightarrow{a_{i1}} C_{i1} \xrightarrow{a_{i2}} C_{i2} \cdots \xrightarrow{a_{ik}} C_{ik})$. By Lemma 5.15, we get that $\mathrm{Prob}(s, W, M) = \mathrm{Prob}(t, W, M)$ i.e. $\alpha'(s)(W, M) =$

$\alpha'(t)(W, M)$ proving that R is a bisimulation on $\langle S, A^*, \alpha' \rangle$ i.e. $s \sim t$ in the $*$-extension $\langle S, A^*, \alpha' \rangle$. $\square$

The same systems of Example 4.4 when each transition is considered as probabilistic with probability 1 show that the $*$-translation Φ^g is also not induced by a natural transformation.

Remark 5.16 The $*$-translation Φ^g together with a subset $\tau \subseteq A$ determines a weak-τ-bisimulation. Thereby the weak-τ-system is

$$\Psi_\tau \circ \Phi^g(\langle S, A, \alpha \rangle) = \Psi_\tau(\langle S, A^*, \alpha' \rangle) = \langle S, A_\tau, \alpha'' \rangle$$

where $\alpha''(s) : \mathcal{P}(A_\tau) \times \mathcal{P}(S) \to [0, 1]$ is given by

$$\alpha''(s) = \ \eta_S^\tau(\alpha'(s)) = \ \mathcal{G}^*\langle h_\tau, id_S \rangle(\alpha'(s)) = \ \alpha'(s) \circ \langle h_\tau^{-1}, id_S \rangle.$$

Hence for $X \subseteq A_\tau$ and $S' \subseteq S$,

$$\alpha''(s)(X, S') = \alpha'(s)(h_\tau^{-1}(X), S') = \alpha'(s)(\bigcup_{w \in X} B_w, S') = \mathrm{Prob}(s, \bigcup_{w \in X} B_w, S'),$$

where, as before, for $w = a_1 \dots a_k \in A_\tau$, B_w is the block $B_w = \tau^* a_1 \tau^* \dots \tau^* a_k \tau^* = h_\tau^{-1}(\{w\})$.

Therefore, from Lemma 5.12 we get that an equivalence relation R is a weak-τ-bisimulation w.r.t. $\langle \Phi^g, \tau \rangle$ on the generative system $\langle S, A, \alpha \rangle$ if and only if $\langle s, t \rangle \in R$ implies that for any collection $(B_i)_{i \in I}$ of blocks and any collection $(C_j)_{j \in J}$ of classes

$$\mathrm{Prob}(s, \bigcup_{i \in I} B_i, \bigcup_{j \in J} C_j) = \mathrm{Prob}(t, \bigcup_{i \in I} B_i, \bigcup_{j \in J} C_j). \tag{27}$$

Sets of the form $\cup_{i \in I} B_i$ will be called *saturated blocks*.

5.3 Correspondence theorem

In this section we recall the original definition of weak bisimulation for generative systems by Baier and Hermanns, and we prove a correspondence theorem, i.e. their weak bisimulation coincides with the weak bisimulation we have obtained in the previous subsection. It is important to note that Baier and Hermanns restrict to finite state systems, in particular they only prove that weak bisimilarity is an equivalence for finite systems. Therefore our result extends the results of Baier and Hermanns to systems with arbitrary state set.

Definition 5.17 [BH97,Bai98,BH99] Let $\langle S, A, \mathrm{P} \rangle$ be a generative system. Let $\tau \in A$ be an invisible action. An equivalence relation $R \subseteq S \times S$ is a weak bisimulation on $\langle S, A, \mathrm{P} \rangle$ if and only if whenever $\langle s, t \rangle \in R$ then for all actions

$a \in A \setminus \{\tau\}$ and for all equivalence classes $C \in S/R$:

$$\mathrm{Prob}(s, \tau^* a \tau^*, C) = \mathrm{Prob}(t, \tau^* a \tau^*, C)$$

and for all $C \in S/R$:

$$\mathrm{Prob}(s, \tau^*, C) = \mathrm{Prob}(t, \tau^*, C).$$

Two states s and t are weakly bisimilar if and only if they are related by some weak bisimulation relation. Notation $s \approx_g t$.

We borrow some properties from Baier and Hermanns, [Bai98,BH99], considering their notion of weak probabilistic bisimulation. We supply proofs since they only consider finite systems.

Proposition 5.18 *Let $\langle S, A, \mathrm{P}\rangle$ be a generative system and let $s \approx_g t$. If R is a weak bisimulation relating s and t, then for all $a_1, \ldots, a_k \in A \setminus \{\tau\}$ and for all classes $C \in S/R$*

$$\mathrm{Prob}(s, \tau^* a_1 \tau^* \ldots \tau^* a_k \tau^*, C) = \mathrm{Prob}(t, \tau^* a_1 \tau^* \ldots \tau^* a_k \tau^*, C).$$

Proof. Let R be a weak bisimulation on $\langle S, A, \mathrm{P}\rangle$ such that $\langle s, t\rangle \in R$. Let $B = \tau^* a_1 \tau^* \ldots \tau^* a_k \tau^*$. We prove 1. by induction on k. For $k \in \{0, 1\}$ the property holds by Definition 5.17. Assume $\mathrm{Prob}(s, B, C) = \mathrm{Prob}(t, B, C)$ for all $C \in S/R$. Let $B' = \tau^* a_1 \tau^* \ldots \tau^* a_k \tau^* a_{k+1} \tau^*$. Then

$$s \xrightarrow{B'} C = \biguplus_{C' \in S/R} s \xrightarrow{B} C' \cdot C' \xrightarrow{\tau^* a_{k+1} \tau^*} C$$

and hence, by Proposition 5.8 and by the hypothesis,

$$\mathrm{Prob}(s, B', C) = \sum_{C' \in S/R} \mathrm{Prob}(s, B, C') \cdot \mathrm{Prob}(C', \tau^* a_{k+1} \tau^*, C) = \mathrm{Prob}(t, B', C).$$

$\square$

Proposition 5.19 *Let $\langle S, A, \mathrm{P}\rangle$ be a generative system and let $s \approx_g t$. Then there exists a weak bisimulation R relating s and t with the property that, for any class $C \in S/R$, $\mathrm{Prob}(s, \tau^*, C) = 1 \Rightarrow s \in C$, i.e., for any two different classes $C_1, C_2 \in S/R$ it holds that $\mathrm{Prob}(C_1, \tau^*, C_2) < 1$.*

Proof. Let R be a weak bisimulation on $\langle S, A, P\rangle$ relating s and t. Define a relation $\rightarrow$ on S/R by

$$C_1 \rightarrow C_2 \iff \mathrm{Prob}(C_1, \tau^*, C_2) = 1$$

and denote by $\leftrightarrow$ the equivalence closure of $\rightarrow$, i.e., $\leftrightarrow = (\rightarrow \cup \leftarrow)^*$.

(a) The relation $\to$ is reflexive and transitive.

Reflexivity is clear, since $\varepsilon \in C \xrightarrow{\tau^*} C$ for any class C. Assume $C_1 \to C_2$ and $C_2 \to C_3$. It is important to note that for any $s \in C_1$,

$$(s \xrightarrow{\tau^*} C_2 \cdot C_2 \xrightarrow{\tau^*} C_3) \uparrow \subseteq \bigcup \{\pi \uparrow \mid \text{first}(\pi) = s, \text{trace}(\pi) \in \tau^*, \text{last}(\pi) \in C_3\}$$

since every cone that contributes to the left-hand-side also contributes to the right-hand-side. Now,

$$\begin{aligned}\text{Prob}((s \xrightarrow{\tau^*} C_2 \cdot C_2 \xrightarrow{\tau^*} C_3) \uparrow) &= \text{Prob}(s \xrightarrow{\tau^*} C_2 \cdot C_2 \xrightarrow{\tau^*} C_3)\\ &= \text{Prob}(s \xrightarrow{\tau^*} C_2) \cdot \text{Prob}(C_2 \xrightarrow{\tau^*} C_3)\\ &= 1\end{aligned}$$

and

$$\text{Prob}(\bigcup \{\pi \uparrow \mid \text{first}(\pi) = s, \text{trace}(\pi) \in \tau^*, \text{last}(\pi) \in C_3\}) = \text{Prob}(C_1 \xrightarrow{\tau^*} C_3).$$

Hence, $1 \leq \text{Prob}(C_1 \xrightarrow{\tau^*} C_3)$ i.e. $\text{Prob}(C_1 \xrightarrow{\tau^*} C_3) = 1$.

(b) Let C_1, C_2, C_3 be different elements of S/R and assume $C_1 \to C_2$. Then either (i) or (ii) holds.

(i) $\forall \pi \in C_1 \xrightarrow{\tau^*} C_3, \exists \pi' \in C_1 \xrightarrow{\tau^*} C_2 \colon \pi' \prec \pi$,
i.e. all τ^* paths from C_1 to C_3 pass C_2.

(ii) $C_3 \to C_2$

Assume $C_1 \to C_2$ and not (i). Let $\pi \in C_1 \xrightarrow{\tau^*} C_3$ be a path that does not pass C_2. Let $s = \text{first}(\pi)$. Since $\text{Prob}(s, \tau^*, C_2) = 1$, also

$$\text{Prob}(\pi \uparrow \cup \biguplus_{\bar{\pi} \in s \xrightarrow{\tau^*} C_2} \bar{\pi} \uparrow) = 1$$

implying that

$$\pi \uparrow \cap \biguplus_{\bar{\pi} \in s \xrightarrow{\tau^*} C_2} \bar{\pi} \uparrow \neq \emptyset$$

i.e., there exists $\bar{\pi} \in s \xrightarrow{\tau^*} C_2$ such that $\pi \uparrow \cap \bar{\pi} \uparrow \neq \emptyset$ which implies that $\pi \prec \bar{\pi}$ (since $\bar{\pi} \prec \pi$ is excluded by not (i)). Now,

$$\pi \uparrow \cup \biguplus_{\bar{\pi} \in s \xrightarrow{\tau^*} C_2} \bar{\pi} \uparrow = \left(\pi \uparrow \cup \biguplus_{\substack{\bar{\pi} \in s \xrightarrow{\tau^*} C_2 \\ \bar{\pi}\uparrow \cap \pi \uparrow \neq \emptyset}} \bar{\pi} \uparrow \right) \sqcup \biguplus_{\substack{\bar{\pi} \in s \xrightarrow{\tau^*} C_2 \\ \bar{\pi}\uparrow \cap \pi \uparrow = \emptyset}} \bar{\pi} \uparrow .$$

Hence,

$$\text{Prob}(\pi \uparrow \cup \biguplus_{\substack{\bar{\pi} \in s \xrightarrow{\tau^*} C_2 \\ \bar{\pi}\uparrow \cap \pi \uparrow \neq \emptyset}} \bar{\pi} \uparrow) + \text{Prob}(\biguplus_{\substack{\bar{\pi} \in s \xrightarrow{\tau^*} C_2 \\ \bar{\pi}\uparrow \cap \pi \uparrow = \emptyset}} \bar{\pi} \uparrow) = 1$$

and

$$\mathrm{Prob}(\biguplus_{\substack{\bar\pi \in s \stackrel{\tau^*}{\to} C_2 \\ \bar\pi\uparrow \cap \pi\uparrow \neq \emptyset}} \bar\pi\uparrow) + \mathrm{Prob}(\biguplus_{\substack{\bar\pi \in s \stackrel{\tau^*}{\to} C_2 \\ \bar\pi\uparrow \cap \pi\uparrow = \emptyset}} \bar\pi\uparrow) = 1$$

i.e.,

$$\mathrm{Prob}(\pi\uparrow \cup \biguplus_{\substack{\bar\pi \in s \stackrel{\tau^*}{\to} C_2 \\ \bar\pi\uparrow \cap \pi\uparrow \neq \emptyset}} \bar\pi\uparrow) = \mathrm{Prob}(\biguplus_{\substack{\bar\pi \in s \stackrel{\tau^*}{\to} C_2 \\ \bar\pi\uparrow \cap \pi\uparrow \neq \emptyset}} \bar\pi\uparrow)$$

and, since for any $\bar\pi \in s \stackrel{\tau^*}{\to} C_2$ with $\bar\pi\uparrow \cap \pi\uparrow \neq \emptyset$ we have (as before) $\pi \prec \bar\pi$ i.e. $\bar\pi\uparrow \subseteq \pi\uparrow$, we get that

$$\mathrm{Prob}(\pi\uparrow) = \mathrm{Prob}(\pi\uparrow \cup \biguplus_{\substack{\bar\pi \in s \stackrel{\tau^*}{\to} C_2 \\ \bar\pi\uparrow \cap \pi\uparrow \neq \emptyset}} \bar\pi\uparrow) = \mathrm{Prob}(\biguplus_{\substack{\bar\pi \in s \stackrel{\tau^*}{\to} C_2 \\ \bar\pi\uparrow \cap \pi\uparrow \neq \emptyset}} \bar\pi\uparrow) \tag{28}$$

Consider the set of paths

$$\Pi = \{\hat\pi = \bar\pi - \pi \mid \pi \cdot \hat\pi = \bar\pi, \bar\pi \in s \stackrel{\tau^*}{\to} C_2, \bar\pi\uparrow \cap \pi\uparrow \neq \emptyset\}.$$

It is easy to see that $\Pi \subseteq \mathrm{last}(\pi) \stackrel{\tau^*}{\to} C_2$ and therefore the set Π is minimal. We have, for any $\hat\pi = \bar\pi - \pi \in \Pi$,

$$\mathrm{Prob}(\hat\pi) = \frac{\mathrm{Prob}(\bar\pi)}{\mathrm{Prob}(\pi)}.$$

Hence

$$\begin{aligned}
\mathrm{Prob}(\Pi) &= \sum_{\hat\pi \in \Pi} \mathrm{Prob}(\hat\pi) \\
&= \frac{\sum_{\substack{\bar\pi \in s \stackrel{\tau^*}{\to} C_2 \\ \bar\pi\uparrow \cap \pi\uparrow \neq \emptyset}} \mathrm{Prob}(\bar\pi)}{\mathrm{Prob}(\pi)} \\
&\stackrel{(*)}{=} \frac{\mathrm{Prob}(\biguplus_{\substack{\bar\pi \in s \stackrel{\tau^*}{\to} C_2 \\ \bar\pi\uparrow \cap \pi\uparrow \neq \emptyset}} \bar\pi\uparrow)}{\mathrm{Prob}(\pi\uparrow)} \\
&\stackrel{(28)}{=} 1
\end{aligned}$$

where $(*)$ holds by the minimality of the set $\{\bar\pi \in s \stackrel{\tau^*}{\to} C_2, \bar\pi\uparrow \cap \pi\uparrow \neq \emptyset\}$. Hence

$$\mathrm{Prob}(C_3 \stackrel{\tau^*}{\to} C_2) \geq \mathrm{Prob}(\Pi) = 1,$$

i.e. $C_3 \to C_2$.

(c) Let C_1, C_2 be different elements of S/R and assume $C_1 \to C_2$. If for $s \in C_1$, $\pi \in s \stackrel{\tau^* a \tau^*}{\to} S$, then there exists $\pi' \in C_1 \stackrel{\tau^*}{\to} C_2$ such that $\pi' \prec \pi$.

Similar argument as for (b) applies here as well. Assume $\pi \in s \stackrel{\tau^* a \tau^*}{\rightarrow} S$. Since $\mathrm{Prob}(s, \tau^*, C_2) = 1$, also

$$\mathrm{Prob}(\pi \uparrow \cup \biguplus_{\bar{\pi} \in s \stackrel{\tau^*}{\rightarrow} C_2} \bar{\pi} \uparrow) = 1$$

implying that

$$\pi \uparrow \cap \biguplus_{\bar{\pi} \in s \stackrel{\tau^*}{\rightarrow} C_2} \bar{\pi} \uparrow \neq \emptyset$$

i.e., there exists $\bar{\pi} \in s \stackrel{\tau^*}{\rightarrow} C_2$ such that $\pi \uparrow \cap \bar{\pi} \uparrow \neq \emptyset$ which implies that $\bar{\pi} \prec \pi$ (since $\pi \prec \bar{\pi}$ is excluded by the form of the traces).

(d) For all $C_1, C_2, C_3 \in S/R$, if $C_1 \rightarrow C_2$ and $C_1 \rightarrow C_3$ then $C_2 \rightarrow C_3$ or $C_3 \rightarrow C_2$.

From (b) we get that either $C_3 \rightarrow C_2$, or each path from C_1 to C_3 with a trace in τ^* passes C_2. Hence, in the later case, we have

$$C_1 \stackrel{\tau^*}{\rightarrow} C_3 \subseteq C_1 \stackrel{\tau^*}{\rightarrow} C_2 \cdot C_2 \stackrel{\tau^*}{\rightarrow} C_3$$

i.e.

$$\mathrm{Prob}(C_1, \tau^*, C_3) \leq \mathrm{Prob}(C_1, \tau^*, C_2) \cdot \mathrm{Prob}(C_2, \tau^*, C_3)$$

which leads to $1 \leq \mathrm{Prob}(C_2, \tau^*, C_3)$ i.e. $C_2 \rightarrow C_3$.

(e) If $C_1 \leftrightarrow C_2$, then there exists C such that $C_1 \rightarrow C$ and $C_2 \rightarrow C$.

We prove this by induction on the length of the sequence of $\rightarrow$ and $\leftarrow$ connecting C_1 and C_2. If $C_1 \rightarrow C_2$, or vice-versa, the statement holds trivially, by reflexivity, with $C = C_2$, or $C = C_1$, in this case. Assume $C_1 \leftrightarrow C_2$ via a sequence of $\rightarrow$ and $\leftarrow$ of length $k+1$. Then there is a C_3 such that $C_1 \leftrightarrow C_3$ via a sequence of $\rightarrow$ and $\leftarrow$ of length k, and, $C_2 \rightarrow C_3$ or $C_3 \rightarrow C_2$. By the inductive hypothesis, there exists C such that $C_1 \rightarrow C$ and $C_3 \rightarrow C$. Now, if $C_2 \rightarrow C_3$, then also, by transitivity, $C_2 \rightarrow C$. If, on the other hand, $C_3 \rightarrow C_2$, then since also $C_3 \rightarrow C$, by (d) we get either $C \rightarrow C_2$ implying $C_1 \rightarrow C_2$, or again $C_2 \rightarrow C$, or both.

(f) Let $F \subseteq S/R$ be a finite set of classes, with the property that for all $C_1, C_2 \in F$, $C_1 \leftrightarrow C_2$. Then there exists a class $C \in S/R$, called a *sink*, such that for all $C' \in F$, $C' \rightarrow C$.

The proof of this property follows simply by (e) and induction on the number of elements in F.

(g) Let $C_0 \in S/R$ be a fixed class such that $U = [C_0]_{\leftrightarrow} \neq \{C_0\}$. Define an equivalence R' on S by determining the set of classes, as

$$S/R' = \{C \in S/R \mid C \not\leftrightarrow C_0\} \cup \{\cup_{C \in U} C\}.$$

Then R' is a weak bisimulation and $R \subset R'$.

We need to prove that for all $a \in A$, all $K_1, K_2 \in S/R'$ and for all $s, t \in K_1$

$$\mathrm{Prob}(s, \tau^*\hat{a}\tau^*, K_2) = \mathrm{Prob}(t, \tau^*\hat{a}\tau^*, K_2)$$

where $\hat{a} = a$ if $a \neq \tau$ and $\hat{\tau} = \varepsilon$, the empty word. We consider four cases:

Case 1. $K_1, K_2 \in S/R$.
The statement holds since R is a weak bisimulation relation.

Case 2. $K_1 \in S/R, K_2 = \cup_{C \in U} C$.
If $U = [C_0]_\leftrightarrow$ contains a sink C for U, i.e. for all $C' \in U$ we have $C' \to C$, we get

$$s \stackrel{\tau^*\hat{a}\tau^*}{\rightarrow} C = s \stackrel{\tau^*\hat{a}\tau^*}{\rightarrow}{}_{\neg K_2} C \uplus \biguplus_{C' \in \ U - \{C\}} s \stackrel{\tau^*\hat{a}\tau^*}{\rightarrow}{}_{\neg K_2} C' \cdot C' \stackrel{\tau^*}{\rightarrow} C$$

and since there are at most countably many R-classes $C' \in U - \{C\}$ for which $s \stackrel{\tau^*\hat{a}\tau^*}{\rightarrow}{}_{\neg K_2} C' \neq \emptyset$, we get

$$\begin{aligned}\mathrm{Prob}(s, \tau^*\hat{a}\tau^*, C) &= \mathrm{Prob}(s, \tau^*\hat{a}\tau^*, C, \neg K_2) \\ &\quad + \sum_{C' \in U - \{C\}} \mathrm{Prob}(s, \tau^*\hat{a}\tau^*, C', \neg K_2) \\ &= \mathrm{Prob}(s, \tau^*\hat{a}\tau^*, K_2)\end{aligned}$$

In the same way we get $\mathrm{Prob}(t, \tau^*\hat{a}\tau^*, C) = \mathrm{Prob}(t, \tau^*\hat{a}\tau^*, K_2)$, i.e.,

$$\mathrm{Prob}(s, \tau^*\hat{a}\tau^*, K_2) = \mathrm{Prob}(t, \tau^*\hat{a}\tau^*, K_2).$$

Note that we only used that U has a sink, and not that it is a whole class of the equivalence relation $\leftrightarrow$.

On the other hand, if U does not contain an R-class which is a sink (and this can only happen for infinite U because of (f)), we use approximation argument. Since there are at most countably many paths outgoing from s, there exists a countable set $U_s \subseteq U$ such that $\mathrm{Prob}(s, \tau^*\hat{a}\tau^*, \cup_{C \in U_s} C) = \mathrm{Prob}(s, \tau^*\hat{a}\tau^*, \cup_{C \in U} C)$. For the same reason, there exists $U_t \subseteq U$, a countable set with the property $\mathrm{Prob}(t, \tau^*\hat{a}\tau^*, \cup_{C \in U_t} C) = \mathrm{Prob}(t, \tau^*\hat{a}\tau^*, \cup_{C \in U} C)$. Taking $U' = U_s \cup U_t$ we get a countable set, such that both

$$\mathrm{Prob}(s, \tau^*\hat{a}\tau^*, \cup_{C \in U'} C) = \mathrm{Prob}(s, \tau^*\hat{a}\tau^*, K_2) \tag{29}$$

and

$$\mathrm{Prob}(t, \tau^*\hat{a}\tau^*, \cup_{C \in U'} C) = \mathrm{Prob}(t, \tau^*\hat{a}\tau^*, K_2). \tag{30}$$

We write $U' = \{C_i \mid i \in \mathbb{N}\}$. We will define a chain of subsets of U in the following way. Put $U_1 = \{C_1\}$ and

$$U_{n+1} = U_n \cup \{C_{n+1}\} \cup \{C^{n+1}\}$$

where $C^{n+1} \in S/R$ is a sink for $U_n \cup \{C^{n+1}\}$. Such a sink exists by (f), and it belongs to U, since U is a $\leftrightarrow$ equivalence class. We have $U_n \subseteq U_{n+1}$ for every natural number n, and also

$$U' \subseteq \cup_{n\in\mathbb{N}} U_n \subseteq U.$$

Next we denote some sets of finite paths. Let

$$\begin{aligned}
\Pi_s^n &= \{\pi \mid \text{first}(\pi) = s, \text{trace}(\pi) \in \tau^*\hat{a}\tau^*, \text{last}(\pi) \in \cup_{C\in U_n} C\}\\
\Pi_s^U &= \{\pi \mid \text{first}(\pi) = s, \text{trace}(\pi) \in \tau^*\hat{a}\tau^*, \text{last}(\pi) \in \cup_{C\in U} C\}\\
\Pi_s^{U'} &= \{\pi \mid \text{first}(\pi) = s, \text{trace}(\pi) \in \tau^*\hat{a}\tau^*, \text{last}(\pi) \in \cup_{C\in U'} C\}
\end{aligned}$$

and similarly we use $\Pi_t^n, \Pi_t^U, \Pi_t^{U'}$. We have

$$\Pi_s^{U'} \subseteq \cup_{n\in\mathbb{N}} \Pi_s^n \subseteq \Pi_s^U$$

and the same holds for t in place of s. Further on, by (29) we have $\text{Prob}(\Pi_s^{U'}) = \text{Prob}(\Pi_s^U)$, hence

$$\text{Prob}(\cup_{n\in\mathbb{N}} \Pi_s^n) = \text{Prob}(s, \tau^*\hat{a}\tau^*, K_2)$$

and also, by (30),

$$\text{Prob}(\cup_{n\in\mathbb{N}} \Pi_t^n) = \text{Prob}(t, \tau^*\hat{a}\tau^*, K_2).$$

Now since $\Pi_s^n \subseteq \Pi_s^{n+1}$ and $\Pi_t^n \subseteq \Pi_t^{n+1}$ we get that

$$\begin{aligned}
\text{Prob}(\cup_{n\in\mathbb{N}} \Pi_s^n) &= \lim_{n\to\infty} \text{Prob}(\Pi_s^n)\\
&= \lim_{n\to\infty} \text{Prob}(s, \tau^*\hat{a}\tau^*, \cup_{C\in U_n} C)\\
&\overset{(*)}{=} \lim_{n\to\infty} \text{Prob}(t, \tau^*\hat{a}\tau^*, \cup_{C\in U_n} C)\\
&= \text{Prob}(\cup_{n\in\mathbb{N}} \Pi_t^n)
\end{aligned}$$

where $(*)$ holds since each U_n is a set of R-classes that contains a sink, which completes the proof of this case.

Case 3. $K_1 = \cup_{C\in U} C$, $K_2 \in S/R'$
Consider $s, t \in K_1$. There exist R-classes C_1 and C_2 such that $s \in C_1$ and $t \in C_2$. We have $C_1 \leftrightarrow C_2$ since $C_1, C_2 \in U$. By (e), there also exists an R-class C such that $C_1 \to C$ and $C_2 \to C$, and moreover $C \in U$, again since U is a $\leftrightarrow$ equivalence class.

If $K_2 = K_1$, then we have

$$\text{Prob}(s, \tau^*, K_2) = \text{Prob}(t, \tau^*, K_2) = 1.$$

If $K_2 \neq K_1$ then we first note that $C \not\to C'$ for any R-class $C' \subseteq K_2$. So, by (b) any τ^* path from C_i to K_2 must pass C, for $i \in \{1, 2\}$. Hence,

$$C_i \xrightarrow{\tau^*} K_2 \subseteq C_i \xrightarrow{\tau^*} C \cdot C \xrightarrow{\tau^*} K_2$$

and moreover, by (b), equality holds, i.e., $C_i \xrightarrow{\tau^*} C = C_i \xrightarrow{\tau^*}{}_{\neg(K_2 \cup C)} C$ since, if a τ^* path from C_i to C passes a class $C' \subseteq K_2$ on the way, then either it was not minimal, i.e. it has a prefix that is also a τ^* path from C_i to C, or $C' \to C$ which is not possible, since $K_2 \neq K_1$. Hence, in this case

$$\begin{aligned}
\mathrm{Prob}(s, \tau^*, K_2) &= \mathrm{Prob}(C_1, \tau^*, C) \cdot \mathrm{Prob}(C, \tau^*, K_2)\\
&= \mathrm{Prob}(C, \tau^*, K_2)\\
&= \mathrm{Prob}(C_2, \tau^*, C) \cdot \mathrm{Prob}(C, \tau^*, K_2)\\
&= \mathrm{Prob}(t, \tau^*, K_2)
\end{aligned}$$

where, the notation $C \xrightarrow{\tau^*} K_2$ as well as $\mathrm{Prob}(C, \tau^*, K_2)$ is justified by Case 2. Next, for $i \in \{1, 2\}$, by (c) we have

$$C_i \xrightarrow{\tau^* a \tau^*} K_2 \subseteq C_i \xrightarrow{\tau^*} C \cdot C \xrightarrow{\tau^* a \tau^*} K_2.$$

Here also equality holds, since no path on the right hand side can have a proper prefix in $C_i \xrightarrow{\tau^* a \tau^*} K_2$. Hence, similar as before,

$$\mathrm{Prob}(s, \tau^* a \tau^*, K_2) = \mathrm{Prob}(C, \tau^* a \tau^*, K_2) = \mathrm{Prob}(t, \tau^* a \tau^*, K_2)$$

and again the notation $C \xrightarrow{\tau^*} K_2$ and $\mathrm{Prob}(C, \tau^*, K_2)$ is justified by Case 2.

(h) Consider the set $\mathcal{W} = \{R' \mid R'$ is a weak bisimulation on $\langle S, A, P\rangle$, $R' \supseteq R\}$, ordered by inclusion. Every ascending chain of $\mathcal{W}$ has an upper bound.

The set $\mathcal{W}$ is not empty since it contains R. Let $\{R_i \mid i \in I\}$ be a chain of elements of $\mathcal{W}$, where I is also a chain of indexes, and $R_i \subseteq R_j$ for $i \leq j$. We show that $\cup_{i \in I} R_i \in \mathcal{W}$.

We first treat the case when I is a countable set, ordered as the natural numbers, $I = \mathbb{N}$, i.e. $\{R_i \mid i \in \mathbb{N}\}$ is a countable chain, with $R_i \subseteq R_{i+1}$. Let $\langle s, t\rangle \in \cup_{i \in \mathbb{N}} R_i$. Then there exists j such that $\langle s, t\rangle \in R_j$, but also $\langle s, t\rangle \in R_n$ for all $n \geq j$. Let $C \in S/\cup_{i \in \mathbb{N}} R_i$. Then $C = \cup_{i \in \mathbb{N}} C_i$ where $C_i \in S/R_i$, and $C_i \subseteq C_{i+1}$ for all i. Consider the sets of paths

$$\begin{aligned}
\Pi_s &= \cup\{\pi \uparrow \mid \mathrm{first}(\pi) = s, \mathrm{trace}(\pi) = \tau^* \hat{a} \tau^*, \mathrm{last}(\pi) \in C\}\\
\Pi_s^i &= \cup\{\pi \uparrow \mid \mathrm{first}(\pi) = s, \mathrm{trace}(\pi) = \tau^* \hat{a} \tau^*, \mathrm{last}(\pi) \in C_i\},\ i \in \mathbb{N}
\end{aligned}$$

Similarly, we use Π_t and Π_t^i, when changing s to t. We have $\Pi_s = \cup_{i \in \mathbb{N}} \Pi_s^i$ and $\Pi_s^i \subseteq \Pi_s^{i+1}$ for all i. Hence,

$$
\begin{aligned}
\mathrm{Prob}(s, \tau^*\hat{a}\tau^*, C) &= \mathrm{Prob}(\Pi_s)\\
&= \mathrm{Prob}(\cup_{i\in\mathbb{N}}\Pi_s^i)\\
&= \lim_{n\to\infty} \mathrm{Prob}(\Pi_s^n)\\
&= \lim_{n\to\infty} \mathrm{Prob}(s, \tau^*\hat{a}\tau^*, C_n)\\
&\stackrel{(*)}{=} \lim_{n\to\infty} \mathrm{Prob}(t, \tau^*\hat{a}\tau^*, C_n)\\
&= \mathrm{Prob}(t, \tau^*\hat{a}\tau^*, C)
\end{aligned}
$$

where $(*)$ holds since for $n \geq j$ we have: $\langle s, t\rangle \in R_n$, C_n is an R_n-class, and R_n is a weak bisimulation.

Next we consider arbitrary chains. Let $\{R_i \mid i \in I\}$ be such. Assume $\langle s, t\rangle \in \cup_{i\in I} R_i$, and let $C \in S/\cup_{i\in I} R_i$. Then $C = \cup_{i\in I} C_i$. Let $I' = \{i \in I \mid \exists\pi\colon \mathrm{first}(\pi) \in \{s, t\}, \mathit{trace}(\pi) \in \tau^*\hat{a}\tau^*, \mathrm{last}(\pi) \in C_i\}$. The set I' is countable. Let $C' = \cup_{i\in I'} C_i$. The set I' can be chosen so that it is ordered as the natural numbers. Let j be the smallest index such that $\langle s, t\rangle \in R_j$, and let $\bar{C}_j$ be the class of R_j containing s and t. Put $\bar{C} = C' \cup \bar{C}_j$. Three situations can occur: 1) $j \in I'$; 2) $j \notin I'$ but there exists $i \in I'$, $i > j$; 3) $j > i$ for all $i \in I'$. In any case we have

$$
\begin{aligned}
\mathrm{Prob}(s, \tau^*\hat{a}\tau^*, C) &= \mathrm{Prob}(s, \tau^*\hat{a}\tau^*, \bar{C})\\
&\stackrel{(*)}{=} \mathrm{Prob}(s, \tau^*\hat{a}\tau^*, \bar{C})\\
&= \mathrm{Prob}(s, \tau^*\hat{a}\tau^*, \bar{C})
\end{aligned}
$$

where, in case of 1) or 2) (*) holds as in the case of a countable ascending chain above, and in case of 3) $\bar{C} = C_j$ and s and t are related by R_j which is a weak bisimulation.

Finally, the statement of the proposition follows from (a) - (h). By (h) and the Zorn's lemma we have that the set

$$\{R' \mid R' \text{ is a weak bisimulation on } \langle S, A, P\rangle,\ R' \supseteq R\}$$

has a maximal element. Let it be $\tilde{R}$. Assume $\tilde{R}$ is not complete, i.e. there exists two different classes $C_1, C_2 \in S/\tilde{R}$ such that $C_1 \to C_2$. Then by (g) we can construct a weak bisimulation $\tilde{R}' \supset \tilde{R}$ which contradicts the maximality of $\tilde{R}$. Hence $\tilde{R}$ is complete i.e. for any two different $C_1, C_2 \in S/\tilde{R}$ we have $\mathrm{Prob}(C_1, \tau^*, C_2) < 1$, and since $R \subseteq \tilde{R}$ it relates s and t which completes the proof. □

We are now able to state and prove the correspondence theorem.

Theorem 5.20 *Let $\langle S, A, \alpha\rangle$ be a generative system. Let $\tau \in A$ be an invisible action and $s, t \in S$ any two states. Then $s \approx_{\{\tau\}} t$ according to Definition 3.3 w.r.t the pair $\langle \Phi^g, \{\tau\}\rangle$ if and only if $s \approx_g t$ according to Definition 5.17.*

The sufficiency part of the theorem holds trivially, having in mind Definition 5.17 and Remark 5.16, equation (27), since τ^* as well as $\tau^*a\tau^*$, for any

$a \in A \setminus \{\tau\}$ is a saturated block and also each R-equivalence class is an R saturated set. Hence $\approx_{\{\tau\}}$ is at least as strong as $\approx_g$. The necessity proof is more involved, and we will split it in several lemmata. Till the end of this subsection we assume that R is a weak bisimulation of a generative system $\langle S, A, \alpha\rangle$ i.e. $\langle S, A, P\rangle$, according to Definition 5.17 satisfying Proposition 5.19, relating s and t.

Lemma 5.21 *For any saturated set $M = \sqcup_{i=1}^n C_i$ consisting of finitely many classes $C_i \in S/R$, for any block $B = \tau^* a_1 \tau^* \dots \tau^* a_k \tau^*$ where $a_1, \dots, a_k \in A \setminus \{\tau\}$ and for any $i \in \{1, \dots, n\}$,*

$$\mathrm{Prob}(s, B, C_i, \neg M) = \mathrm{Prob}(t, B, C_i, \neg M).$$

Proof. We use induction on n, the number of classes that M contains. For $n = 1$ the property is simply Proposition 5.18. Assume $\mathrm{Prob}(s, B, C_i, \neg M) = \mathrm{Prob}(t, B, C_i, \neg M)$ for any R - saturated set M being a union of less than n classes, and any class $C_i \subseteq M$. Let M be an R- saturated set which is a union of n classes, i.e. $M = \sqcup_{i=1}^n C_i$ for some $C_i \in S/R$. We use the following notation, for $i \in \{1, \dots, n\}$ and $j \in \{1, \dots, i-1, i+1, \dots, n\}$.

$$V_i = \mathrm{Prob}(s, B, C_i) \stackrel{5.18}{=} \mathrm{Prob}(t, B, C_i)$$

$$G_i^j = \mathrm{Prob}(s, B, C_j, \neg \sqcup_{k=1,k\neq i}^n C_k) \stackrel{IH}{=} \mathrm{Prob}(t, B, C_j, \neg \sqcup_{k=1,k\neq i}^n C_k)$$

$$T_i^j = \mathrm{Prob}(C_j, \tau^*, C_i)$$

$$H_i^j = \mathrm{Prob}(C_i, \tau^*, C_j, \neg \sqcup_{k=1,k\neq i}^n C_k)$$

Consider the series $\sum_{k\geq 0} a_k$ for

$$a_{2k} = V_i \cdot \left(\sum_{j=1,j\neq i}^n H_i^j \cdot T_i^j \right)^k, \quad a_{2k+1} = -\left(\sum_{j=1,j\neq i}^n G_i^j \cdot T_i^j \right) \left(\sum_{j=1,j\neq i}^n H_i^j \cdot T_i^j \right)^k.$$

Note that $\sum_{k\geq 0} a_{2k}$ is a geometric series, and $\sum_{k\geq 0} a_{2k+1}$ as well, with the same ratio $\rho = \sum_{j=1,j\neq i}^n H_i^j \cdot T_i^j$. Let $T_i = \max_{j=1,j\neq i}^n T_i^j$. By Proposition 5.19, $T_i^j < 1$ for all $j \neq i$ and therefore $T_i < 1$. Furthermore, note that

$$\sum_{j=1,j\neq i}^n H_i^j = \mathrm{Prob}(C_i, \tau^*, \sqcup_{j=1,j\neq i}^n C_j) \leq 1.$$

Hence,

$$\rho = \sum_{j=1,j\neq i}^n H_i^j \cdot T_i^j \leq T_i \cdot \sum_{j=1,j\neg i} H_i^j \leq T_i < 1$$

i.e., both geometric series are convergent. Moreover, they are absolutely convergent. Therefore the series $\sum_{k\geq 0} b_k$ for $b_k = a_{2k} + a_{2k+1}$ is absolutely convergent which means that $\sum_{k\geq 0} a_k$ is as well, being just a rearrangement of the elements of $\sum_{k\geq 0} b_k$. Note that

$$\sum_{k\geq 0} a_k = \frac{V_i - \sum_{j=1,j\neq i}^{n} G_i^j \cdot T_i^j}{1 - \sum_{j=1,j\neq i}^{n} H_i^j \cdot T_i^j}$$

and this value does not depend on the starting state s. We will prove that $\sum_{k\geq 0} a_k = \text{Prob}(s, B, C_i, \neg M)$ which is enough to conclude that $\text{Prob}(s, B, C_i, \neg M) = \text{Prob}(t, B, C_i, \neg M)$. For this purpose we give meaning to a_k using the results of Subsection 5.1. We first denote some sets of finite paths. Let

$$\Pi_{2k} = s \xrightarrow{B} C_i \cdot (C_i \xrightarrow{\tau^*} \sqcup_{m=1,m\neq i}^{n} C_m \xrightarrow{\tau^*} C_i)^k$$

$$\Pi_{2k+1} = s \xrightarrow{B} \sqcup_{m=1,m\neq i}^{n} C_m \xrightarrow{\tau^*} C_i \cdot (C_i \xrightarrow{\tau^*} \sqcup_{m=1,m\neq i}^{n} C_m \xrightarrow{\tau^*} C_i)^k.$$

By Proposition 5.8, Corollary 5.9, the definition of a_i, with a help of an inductive argument one obtains that for any $i \geq 0$

$$a_i = (-1)^i \cdot \text{Prob}(\Pi_i).$$

In order to prove that $\sum_{k\geq 0} a_k = \text{Prob}(s, B, C_i, \neg M)$ we define a function $\omega : s \xrightarrow{B} S \rightarrow \{1,2\}^*$. The function ω will, in a sense, trace the classes that a path visits with a word in B. Some auxiliary functions will be needed for the definition of ω. Let $\tilde{\omega} : s \xrightarrow{B} S \rightarrow \{1,2\}^*$ be defined by

$$\tilde{\omega}(\pi \cdot \text{last}(\pi) \xrightarrow{a} t) = \begin{cases} 1 & t \in C_i, \pi \notin s \xrightarrow{B} S \\ 2 & t \in M \setminus C_i, \pi \notin s \xrightarrow{B} S \\ \epsilon & t \notin M, \pi \notin s \xrightarrow{B} S \\ \tilde{\omega}(\pi) \cdot 1 & t \in C_i, \pi \in s \xrightarrow{B} S \\ \tilde{\omega}(\pi) \cdot 2 & t \in M \setminus C_i, \pi \in s \xrightarrow{B} S \\ \tilde{\omega}(\pi) & t \notin M, \pi \in s \xrightarrow{B} S \end{cases}$$

and if $\epsilon \in s \xrightarrow{B} S$, then $\tilde{\omega}(\epsilon) = \epsilon$.

Let $d : \{1,2\}^* \rightarrow \{1,2\}^*$ and $d' : \{1,2\}^* \rightarrow \{1,2\}^*$ be defined in the following way, for $u, v \in \{1,2\}^*$ and $x, y \in \{1,2\}$.

$$d(u \cdot x) = \begin{cases} d(u) \cdot x & u = v \cdot x \\ d'(u) \cdot x & u = v \cdot y, y \neq x \end{cases}$$

$$d'(u \cdot x) = \begin{cases} d'(u) & u = v \cdot x \\ d'(u) \cdot x & u = v \cdot y, y \neq x \end{cases}$$

We put $\omega = d \circ \tilde{\omega}$. We can explain the definition of the maps d, $\tilde{\omega}$ and ω as follows. The map $\tilde{\omega}$ takes a path with a trace in B and encodes the sequence of the classes that are visited by the path after a word in B has already been performed. The encoding is 1 if the class under consideration, C_i, has been visited and 2 if any other class from M has been visited, there is no record of classes outside M. Then the map d removes adjacent multiple occurrences of 1 and 2 in the word obtained by $\tilde{\omega}$, except for the multiple occurrences at the end of the word. Basically, the map d is computed by the normal algorithm $\{112 \to 12, 221 \to 21\}$. It is important to note the following.

$$\omega^{-1}(\{1, 21\}) = \ s \xrightarrow{B} C_i, \quad \omega^{-1}(\{1\}) = \ s \xrightarrow{B}_{\neg M} C_i$$

By the definition of ω we easily get that

$$\omega^{-1}(\{1, 21\}) = \omega^{-1}(\{1\}) \uplus \omega^{-1}(\{21\}).$$

A more careful inspection shows that

$$\omega^{-1}(\{21\}) \uplus \left(\uplus_{j=1, j \neq i}^{n} \omega^{-1}(\{1\}) \cdot C_i \xrightarrow{\tau^*}_{\neg M \setminus C_i} C_j \xrightarrow{\tau^*} C_i \right)$$

$$= \uplus_{j=1, j \neq i}^{n} s \xrightarrow{B}_{\neg M \setminus C_i} C_j \xrightarrow{\tau^*} C_i.$$

Therefore, using again Proposition 5.8 and Corollary 5.9 we get

$$\begin{aligned} \mathrm{Prob}(\omega^{-1}(\{1\})) &= \mathrm{Prob}(\omega^{-1}(\{1, 21\})) - \mathrm{Prob}(\omega^{-1}(\{21\})) \\ &= V_i - \left(\sum_{j=1, j \neq i}^{n} G_i^j \cdot T_i^j - \mathrm{Prob}(\omega^{-1}(\{1\})) \cdot \sum_{j=1, j \neq i}^{n} H_i^j \cdot T_i^j \right) \\ &= a_0 + a_1 + \mathrm{Prob}(\omega^{-1}(\{1\})) \cdot \rho \end{aligned}$$

i.e.,

$$\mathrm{Prob}(s, B, C_i, \neg M) = a_0 + a_1 + \mathrm{Prob}(s, B, C_i, \neg M) \cdot \rho.$$

Hence, for all $n \geq 0$

$$\mathrm{Prob}(s, B, C_i, \neg M) = \sum_{k=0}^{n-1} (a_0 + a_1) \cdot \rho^k + \mathrm{Prob}(s, B, C_i, \neg M) \cdot \rho^n$$

and since $\lim_{n \to \infty} \mathrm{Prob}(s, B, C_i, \neg M) \cdot \rho^n = 0$, we get

$$\mathrm{Prob}(s, B, C_i, \neg M) = \lim_{n \to \infty} \sum_{k=0}^{n} (a_0 + a_1) \cdot \rho^k = \sum_{k \geq 0} a_k$$

which completes the proof. $\square$

Let us remark that another proof of the above property can be obtained using the invertibility of the matrix $M_{i,j} = \mathrm{Prob}(C_i, \tau^*, C_j)$ for $i, j \in \{1, \ldots, n\}$, as it is used in [Bai98,BH99]. However, we find the proof method applied above interesting. Next we extend the property to arbitrary R-saturated sets.

Lemma 5.22 *For any R-saturated set M, for any block $B = \tau^* a_1 \tau^* \ldots \tau^* a_k \tau^*$ where $a_1, \ldots, a_k \in A \setminus \{\tau\}$ and for any class $C \subseteq M$*

$$\mathrm{Prob}(s, B, C, \neg M) = \mathrm{Prob}(t, B, C, \neg M).$$

Proof. We will show that we can assume that M contains at most countably many classes. Let S' be the set of states that are reachable from s by a finite path. This set is at most countable since each finite path contributes to S' with finitely many states, and there are at most countably many paths starting in s according to Lemma 5.1. Let M_s be the smallest R-saturated set containing $S' \cap M$. Since $S' \cap M$ is at most countable, the set M_s contains at most countably many classes and $\mathrm{Prob}(s, B, C_i, \neg M) = \mathrm{Prob}(s, B, C_i, \neg M_s)$. In the same way we get a saturated set M_t containing at most countably many classes such that $\mathrm{Prob}(t, B, C_i, \neg M) = \mathrm{Prob}(t, B, C_i, \neg M_t)$. Then $M' = M_s \cup M_t$ is a saturated set containing at most countably many classes and

$$\mathrm{Prob}(s, B, C_i, \neg M') = \mathrm{Prob}(s, B, C_i, \neg M),$$

$$\mathrm{Prob}(t, B, C_i, \neg M') = \mathrm{Prob}(t, B, C_i, \neg M).$$

So, assume $M = \sqcup_{j \geq 0} C_j$. Note that

$$s \xrightarrow{B}_{\neg M} C = \bigcap_{C' \subseteq M} s \xrightarrow{B}_{\neg C'} C.$$

We use the following simple property from measure theory. If μ is a probability measure on some set and if $A = \cap_{n \in \mathbb{N}} A_n$ is a measurable set which is a countable intersection of measurable sets, then $\mu(A) = \inf\{\mu(\cap_{i \in I} A_i) \mid I \subseteq \mathbb{N}, I \text{ finite }\}$. Hence,

$$\begin{aligned}
&\mathrm{Prob}(s, B, C, \neg M)\\
&\quad = \inf\{\mathrm{Prob}(\cap_{C' \subseteq I_M} s \xrightarrow{B}_{\neg C'} C) \mid I_M \subseteq M, I_M \text{ "finite"}\}\\
&\quad = \inf\{\mathrm{Prob}(s, B, C, \neg I_M) \mid I_M \subseteq M, I_M \text{"finite"}\}\\
&\quad \overset{5.21}{=} \inf\{\mathrm{Prob}(t, B, C, \neg I_M) \mid I_M \subseteq M, I_M \text{ "finite"}\}\\
&\quad = \mathrm{Prob}(t, B, C, \neg M)
\end{aligned}$$

where "finite" means a saturated set containing finitely many classes. □

By Lemma 5.22, noting that $\mathrm{Prob}(s, B, M) = \mathrm{Prob}(s, B, \sqcup_{i \in I} C_i) = \sum_{i \in I} \mathrm{Prob}(s, B, C_i, \neg M)$ we get the following property.

Corollary 5.23 *For any R-saturated set M, for any block $B = \tau^* a_1 \tau^* \dots \tau^* a_k \tau^*$ where $a_1, \dots, a_k \in A \setminus \{\tau\}$*

$$\mathrm{Prob}(s, B, M) = \mathrm{Prob}(t, B, M).$$

□

We proceed to saturated blocks. Again we first treat saturated blocks containing finitely many blocks and then extend to arbitrary saturated blocks.

Lemma 5.24 *For any R-saturated set M and for any saturated block $W = \sqcup_{j=1}^n B_j$ containing finitely many blocks*

$$\mathrm{Prob}(s, W, M) = \mathrm{Prob}(t, W, M).$$

Proof. Note that

$$\mathrm{Prob}(s, W, M) = \sum_{i=1}^{n} \mathrm{Prob}(s, B_i, \neg W, M)$$

since

$$S \xrightarrow{W} M = \biguplus_{i=1}^{n} s \xrightarrow{B_i}_{\neg W} M,$$

and also

$$\mathrm{Prob}(s, B_i, \neg W, M) = \sum_{j:\ C_j \in M} \mathrm{Prob}(s, B_i, \neg W, C_j, \neg M)$$

for similar reasons, where the last equation holds since we can assume that M contains at most countably many classes. Hence it is enough to prove that

$$\mathrm{Prob}(s, B_i, \neg W, C_j, \neg M) = \mathrm{Prob}(t, B_i, \neg W, C_j, \neg M)$$

for any B_i, $i \in \{1, \dots, n\}$ and any class $C_j \subseteq M$. For any i, let $w_i \in A \setminus \{\tau\}^*$, $w_i = a_{i1} \dots a_{ik_i}$ be the word such that $B_i = B_{w_i} = \tau^* a_{i1} \tau^* \cdots \tau^* a_{ik_i} \tau^*$. The prefix ordering on the set of words $\{w_1, \dots, w_n\}$ induces an ordering on the set of blocks $\{B_1, \dots, B_n\}$ given by $B_i \prec B_j$ if and only if $w_i \prec w_j$. If $B_i \prec B_j$, by B_{j-i} we denote the block corresponding to w_{j-i}, the unique word satisfying $w_i \cdot w_{j-i} = w_j$. We are going to prove, by induction on the number of elements in the set $\{i \in \{1, \dots, n\} \mid B_i \prec B_j\}$ that

$$s \xrightarrow{B_j}_{\neg M} C = s \xrightarrow{B_j}_{\substack{\neg W \\ \neg M}} C \uplus \left(\biguplus_{B_i \prec B_j} \biguplus_{C' \subseteq M} s \xrightarrow{B_i}_{\substack{\neg W \\ \neg M}} C' \xrightarrow{B_{j-i}}_{\neg M} C \right) \tag{31}$$

where $C' \subseteq M$ is a class. First of all we have to make sure that the right hand side of the equation is well defined, i.e. that the unions are really disjoint and minimal. By the definition of the involved sets of paths a careful inspection shows that it is indeed the case. It is rather obvious that the right hand side is contained in the left hand side since all the paths of the right hand side do start in s, have a trace in B_j and end up in C, without reaching M before with a prefix whose trace is also in B_j. For the opposite inclusion we use inductive argument. Assume B_j has no (strict) prefixes in $\{B_1, \dots, B_n\}$. Then the equation becomes $s \stackrel{B_j}{\rightarrow}_{\neg M} C = s \stackrel{B_j}{\rightarrow}_{\substack{\neg W \\ \neg M}} C$ and it holds since no path which has a trace in B_j can have a strict prefix with a trace in W. For the inductive step, assume $\pi \in s \stackrel{B_j}{\rightarrow}_{\neg M} C$ and $\pi \notin s \stackrel{B_j}{\rightarrow}_{\substack{\neg W \\ \neg M}} C$. This means that π has a prefix that has a trace in $\cup_{i=1}^{n} B_i$ and ends in M. So, $\pi \in s \stackrel{B_k}{\rightarrow} C' \stackrel{B_{j-k}}{\rightarrow}_{\neg M} C$ for some k and for some class $C' \subseteq M$. We want to show that $\pi \in \biguplus_{B_i \prec B_j} \biguplus_{C' \subseteq M} s \stackrel{B_i}{\rightarrow}_{\substack{\neg W \\ \neg M}} C' \stackrel{B_{j-i}}{\rightarrow}_{\neg M} C$. We can assume that $\pi \in s \stackrel{B_k}{\rightarrow}_{\neg M} C' \stackrel{B_{j-k}}{\rightarrow}_{\neg M} C$ by taking C' to be the first class of M that π hits having performed a trace in B_k. Now B_k, being a prefix of B_j, has less prefixes than B_j and therefore either

$$\pi \in s \stackrel{B_k}{\rightarrow}_{\substack{\neg W \\ \neg M}} C' \stackrel{B_{j-k}}{\rightarrow}_{\neg M} C$$

or there exist $r \in \{1, \dots, n\}$ and a class $C'' \subseteq M$ such that

$$\pi \in s \stackrel{B_r}{\rightarrow}_{\substack{\neg W \\ \neg M}} C'' \stackrel{B_{k-r}}{\rightarrow}_{\neg M} C' \stackrel{B_{j-k}}{\rightarrow}_{\neg M} C$$

i.e. $\pi \in s \stackrel{B_r}{\rightarrow}_{\substack{\neg W \\ \neg M}} C'' \stackrel{B_{j-r}}{\rightarrow}_{\neg M} C$, which completes the proof of equation (31). Now, by the same inductive argument we get: if B_j has no proper prefixes than

$$\begin{aligned} \mathrm{Prob}(s, B_j, \neg W, C, \neg M) &= \mathrm{Prob}(s, B_j, C, \neg M) \\ &= \mathrm{Prob}(t, B_j, C, \neg M) \\ &= \mathrm{Prob}(t, B_j, \neg W, C, \neg M). \end{aligned}$$

Assume that $\mathrm{Prob}(s, B_i, \neg W, C, \neg M) = \mathrm{Prob}(t, B_i, \neg W, C, \neg M)$ for all $B_i \prec B_j$. Then by (31) and by Proposition 5.8 we get

$$\begin{aligned}
&\mathrm{Prob}(s, B_j, \neg W, C, \neg M)\\
&\quad= \mathrm{Prob}(s, B_j, C, \neg M)\\
&\qquad - \sum_{B_i \prec B_j} \sum_{C' \subseteq M} \mathrm{Prob}(s, B_i, \neg W, C', \neg M) \cdot \mathrm{Prob}(C', B_{j-i}, C, \neg M)\\
&\quad\overset{(IH)}{=} \mathrm{Prob}(t, B_j, C, \neg M)\\
&\qquad - \sum_{B_i \prec B_j} \sum_{C' \subseteq M} \mathrm{Prob}(t, B_i, \neg W, C', \neg M) \cdot \mathrm{Prob}(C', B_{j-i}, C, \neg M)\\
&\quad= \mathrm{Prob}(t, B_j, \neg W, C, \neg M)
\end{aligned}$$

which completes the proof. □

Lemma 5.25 *For any R-saturated set M and for any saturated block W*

$$\mathrm{Prob}(s, W, M) = \mathrm{Prob}(t, W, M).$$

Proof. We first consider the countable case. Let $W = \sqcup_{n\in\mathbb{N}} B_n$. Let

$$\Pi_n^s = \{\pi \mid \mathrm{first}(\pi) = s, \mathrm{last}(\pi) \in M, \mathrm{trace}(\pi) \in B_n\}$$

$$\Pi_n^t = \{\pi \mid \mathrm{first}(\pi) = t, \mathrm{last}(\pi) \in M, \mathrm{trace}(\pi) \in B_n\}.$$

Then

$$\begin{aligned}
\mathrm{Prob}(s, W, M) &= \mathrm{Prob}(s, \sqcup_{n\in\mathbb{N}} B_n, M)\\
&= \mathrm{Prob}((\cup_{n\in\mathbb{N}} \Pi_n^s)\downarrow)\\
&= \mathrm{Prob}(\cup_{n\in\mathbb{N}} \Pi_n^s)\\
&\overset{(*)}{=} \sup\{\mathrm{Prob}(\cup_{i\in I} \Pi_i^s) \mid I \subseteq \mathbb{N}, I \text{ finite }\}\\
&= \sup\{\mathrm{Prob}(s, W_I, M) \mid W_I = \sqcup_{i\in I} B_i, I \text{ finite }\}\\
&= \sup\{\mathrm{Prob}(t, W_I, M) \mid W_I = \sqcup_{i\in I} B_i, I \text{ finite }\}\\
&= \mathrm{Prob}(t, W, M).
\end{aligned}$$

where the equality $(*)$ holds because of the following simple property from measure theory. Let μ be a measure on some set, and let $A = \cup_{n\in\mathbb{N}} A_n$ be a measurable set which is a countable union of measurable sets. Then $\mu(A) = \sup\{\mu(\cup_{i\in I} A_i) \mid I \subseteq \mathbb{N}, I \text{ finite}\}$.

If $W = \sqcup_{i\in I} B_i$ contains arbitrary many blocks then there exists a countable index set $I_s \subseteq I$ and a saturated set $W_s = \sqcup_{i\in I_s} B_i$ such that $\mathrm{Prob}(s, W, M) = \mathrm{Prob}(s, W_s, M)$ using Lemma 5.1. For the same reason, there exists a countable index set $I_t \subseteq I$ and a corresponding saturated set $W_t = \sqcup_{i\in I_t} C_i$ with $\mathrm{Prob}(t, W, M) = \mathrm{Prob}(t, W_t, M)$. Hence $\mathrm{Prob}(s, W, M) = \mathrm{Prob}(s, W_s \cup W_t, M) = \mathrm{Prob}(t, W_s \cup W_t, M) = \mathrm{Prob}(t, W, M)$ since $W_s \cup W_t$ is countable, and that case we have already proven. □

Note that Lemma 5.25 proves the necessity part of the correspondence Theorem 5.20.

6 Conclusions

In this paper we have proposed a coalgebraic definition of weak bisimulation for action-type systems. For its justification we have considered the case of the familiar labelled transition systems and of generative probabilistic systems and have argued that the coalgebraic notion coincides with the concrete definitions. Additionally, the paper also comprises a few other, smaller contributions.

This paper follows an earlier work jointly with Falk Bartels [BSV03,BSV]. In Section 2 we have discussed a general method for obtaining correspondence results for coalgebraic versus concrete bisimulations. The main idea is to tie up the reformulation of coalgebraic bisimulation in terms of the lifted bisimulation relation $\equiv_{\mathcal{F},R}$ and the pullback of a particular cospan (cf. Lemma 2.11).

Our handling of probabilistic distributions avoids restricting the cardinality of the support set, a fact of some technical interest. The results hold for arbitrary discrete distributions captured by the functor $\mathcal{D}$ of Section 2. Although we do not impose cardinality restrictions on the state spaces considered, generative probabilistic system are discrete in nature. The work of Baier and Hermanns treats finite systems only, also because of the algorithmic considerations addressed [BH97,BH99], a matter that we do not touch upon here. The formulations, both concrete and coalgebraic, as used in the present paper extend the work of Baier and Hermanns in the sense that we do not impose this restriction.

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