Proofs with ∃-introduction and ∃-elimination are unnecessarily long and cumbersome…

There are alternatives!

Proving an existential quantification

To prove

$\exists x[x \in \mathbb{Z} : x^3 - 2x - 8 \geq 0]$

Proof

It suffices to find a witness, i.e., an $x \in \mathbb{Z}$ satisfying

$$x^3 - 2x - 8 \geq 0.$$

$x = 3$ is a witness, since $3 \in \mathbb{Z}$ and $3^3 - 2 \cdot 3 - 8 = 13 \geq 0$

Conclusion: $\exists x[x \in \mathbb{Z} : x^3 - 2x - 8 \geq 0]$.

also $x = 5$ is a witness…

Alternative ∃ introduction

How do we prove an existential quantification?

by finding a witness

∃∗-introduction

...

(k) P(a)

...

(l) Q(a)

...

{∃∗-intro on (k) and (l)}

(m) ∃x [P(x) : Q(x)]

strategy: wait until a witness object appears

does not always work

(k < m, l < m)

Using an existential quantification

We know $\exists x[x \in \mathbb{R} : a - x < 0 < b - x]$

We can declare an $x \in \mathbb{Z}$ (a witness) such that

$$a - x < 0 < b - x$$

and use it further in the proof. For example:

From $a - x < 0$, we get $a < x$.
From $b - x > 0$, we get $x < b$.
Hence, $a < b$.

Alternative ∃ elimination

How do we use an existential quantification in a proof?

we pick a witness

∃∗-elimination

|| ||

(k) $\exists x\ [P(x) : Q(x)]$

|| ||

{∃∗-elim on (k)}

(m) Pick x with P(x) and Q(x)

x must be new!

(k < m)

time for an example!

Back to
Naive Set Theory
Relations

Product of multiple sets

Direct product (Kartesisches Produkt)

$$A \times B = \{(x,y) \mid x \in A \text{ and } y \in B\}$$

ordered pairs

$(A \times B) \times C \neq A \times (B \times C)$

Therefore, we define

$$A \times B \times C = \ldots \text{ and } y \in B \text{ and } z \in C\}$$

if $A_i = A$ for all i, then the product is denoted A^n

In general, for sets $A_1, A_2, \ldots, A_n$ with $n \geq 1$,

sequence of length n

$$A_1 \times A_2 \times \ldots \times A_n = \prod_{1 \leq i \leq n} A_i = \{(x_1,x_2,\ldots,x_n) \mid x_i \in A_i \text{ for } 1 \leq i \leq n\}$$

Relations

Def. If A and B are sets, then any subset $R \subseteq A \times B$ is a (binary) relation between A and B

similarly, unary relation (subset), n-ary relation...

Def. R is a relation on A if $R \subseteq A \times A$

some relations are special

Special relations

A relation $R \subseteq A \times A$ is:

reflexive	iff	for all $a \in A$, $(a,a) \in R$
symmetric	iff	for all $a,b \in A$, if $(a,b) \in R$, then $(b,a) \in R$
transitive	iff	for all $a,b,c \in A$, if $(a,b) \in R$ and $(b,c) \in R$, then $(a,c) \in R$
irreflexive	iff	for all $a \in A$, $(a,a) \notin R$
antisymmetric	iff	for all $a,b \in A$, if $(a,b) \in R$ and $(b,a) \in R$ then $a = b$
asymmetric	iff	for all $a,b \in A$, if $(a,b) \in R$, then $(b,a) \notin R$
total	iff	for all $a,b \in A$, $(a,b) \in R$ or $(b,a) \in R$

(infix) notation aRb for $(a,b) \in R$

Special relations

A relation R on A, i.e., R ⊆ A x A is:

equivalence	iff	R is reflexive, symmetric, and transitive
partial order	iff	R is reflexive, antisymmetric, and transitive
strict order	iff	R is irreflexive and transitive
preorder	iff	R is reflexive and transitive
total (linear) order	iff	R is a total partial order

Obvious properties

1. Every partial order is a preorder.

2. Every total order is a partial order.

3. Every total order is a preorder.

4. If $R \subseteq A \times A$ is a relation such that there are $a, b \in A$ with

$$a \neq b, (a,b) \in R \text{ and } (b,a) \in R,$$

   then R is not a partial order, nor a total order, nor a strict order.